The modular geometry of Random Regge Triangulations

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Abstract

We show that the introduction of triangulations with variable connectivity and fluctuating egde-lengths (Random Regge Triangulations) allows for a relatively simple and direct analyisis of the modular properties of 2 dimensional simplicial quantum gravity. In particular, we discuss in detail an explicit bijection between the space of possible random Regge triangulations (of given genus g and with N_0 vertices) and a suitable decorated version of the (compactified) moduli space of genus g Riemann surfaces with N_0 punctures $\overline{\mathfrak{M}}_{g,N_0}$. Such an analysis allows us to associate a Weil-Petersson metric with the set of random Regge triangulations and prove that the corresponding volume provides the dynamical triangulation partition function for pure gravity.

PACS: 04.60.Nc, 04.60.K

Keywords: Regge calculus, dynamical triangulations theory, 2D quantum gravity.

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1 Introduction

In this paper we discuss some aspects of 2-dimensional simplicial quantum gravity by using surfaces endowed with triangulations with variable connectivity and fluctuating egde-lengths. Such piecewise-linear (PL) surfaces are not proper Regge triangulations, since their adjacency matrix is not a priori fixed, nor they are dynamical triangulations, since they are generated by glueing triangles which are not, in general, equilateral. Lacking better names, we call them Random Regge Triangulations. They inherit most of the nice features of their parent triangulations and put in a more proper perspective some of the pathologies of Regge triangulations which have often been misjudged. For instance, we show that most of these pathologies have a modular origin and are naturally related to the Weil-Petersson geometry of the (compactified) moduli space of genus g Riemann surfaces with N_0 punctures $\overline{\mathfrak{M}}_{g,N_0}$, (the number of punctures N_0 is the number of vertices of the triangulations). As a matter of fact, the main feature of random Regge triangulations is their allowing for a relatively simple and direct analysis of the modular properties of 2 dimensional simplicial quantum gravity. Usually, any such analysis requires techniques from the theory of integrable system, matrix models, and conformal field theory. This interdisciplinariety is one of the reason that motivates the success of two-dimensional quantum gravity, however it tends to relegate simplicial quantum gravity to the ancillary role of a regularization scheme. In such a sense, it is important to establish a connection between moduli space theory and simplicial quantum gravity which is more directly a consequence of the use of simplicial methods, thus making their role more foundational. In order to achieve such an objective, we explicitly map random Regge triangulations into the theory of singular Euclidean structures. These latter are represented by closed surfaces endowed with a metric which is locally isometric to the euclidean plane except for a finite number of cone singularities. Such a mapping allows for a rather straightforward bijection between the space of possible random Regge triangulations (of given genus g and with N_0 vertices) and a suitable decorated version of the moduli space $\overline{\mathfrak{M}}_{g,N_0}$. The construction we present here is partly related to the geometrical setup of the Witten-Kontsevich model and the combinatorial parametrization of $\overline{\mathfrak{M}}_{g,N_0}$ in terms of ribbon graphs theory. However, even if strongly based on ribbon graph theory, our approach requires a careful treatment of the role of the deficit angles and the development of a technique for relating the Regge geometry of the triangulations to the modular parameters of $\overline{\mathfrak{M}}_{g,N_0}$. The main result of our analysis is the explicit association of a Weil-Petersson metric to a Regge triangulation. With such a metric at our disposal, we can formally evaluate the Weil-Petersson volume over the space of all random Regge triangulations. By exploiting a recent result of Manin and Zograf [1], it is then an easy matter to show that such a volume provides the (large N_0 asymptotics of the) dynamical triangulation partition function for pure gravity.

Let us briefly summarize the content of the paper. In Section 2, after providing a few basic definitions, we describe the interplay between (random) Regge triangulations and the theory of singular Euclidean structures. To some extent

this is the underlying rationale of the paper, allowing us to develop a dictionary between piecewise-linear surface geometry and standard complex analytical surface theory. Subsections 2.1, and 2.2 elaborate on this and related matters, such as the elegant Troyanov formulation of the Gauss-Bonnet theorem for triangulated surfaces and its implications. In particular, in subsection 2.3 we discuss the analytical meaning of zero-deficit angle degenerations in Regge calculus. Moreover, in subsection 2.4 we explicitly relate the Regge deficit angles to suitable moduli parameters in the complex uniformizations of the stars of the vertices of the triangulation. This result features prominently in the main applications of the paper, since it motivates the analysis of the geometry of Regge triangulations from the point of view of moduli theory. Clearly such an analysis must come to terms with the well-known correspondence between ribbon graph theory and combinatorial cell decompositions of the moduli space $\overline{\mathfrak{M}}_{g,N_0}$ which is at the basis of the Witten-Kontsevich model. In such a connection we discuss, in section 3, how we can naturally associate a ribbon graph with the 1-skeleton of a(random) Regge polytope. We also elaborate on the stratified orbifold structure of the space of random Regge polytopes. In subsection 3.2 we show that these orbifold strata are naturally parametrized by dynamical triangulations. It is worthwhile stressing that such results can be seen as a rather elementary consequence of the remarkable analysis of the space of ribbon graphs due to M. Mulase and M. Penkawa [2]. Our debt to the paper [2] extends further in section 4, where we specialize some of their results on ribbon graphs theory in order to provide an explicit correspondence between (random)Regge triangulations an N_0-pointed Riemann surfaces (proposition 3, subsection 4.1). We come to full circle with moduli space theory when, by exploiting such a correspondence, we discuss deformations of Regge triangulations (subsection4.2). In such a setting, by examining the analytical aspects of Regge deformations, one is naturally led to consider the Weil-Petersson measure on the space of Regge polytopes. Among other things, such a construction further elucidates the deep geometrical nature of the degenerations in Regge calculus. Finally, in subsection 4.3 we explicitly show that the (asymptotic) evaluation of the Weil-Petersson volume over the space of random Regge polytopes is strictly related to the canonical partition function of dynamical triangulation theory.

2 Triangulated surfaces and Polytopes

Let T denote a 2-dimensional simplicial complex with underlying polyhedron $|T|$ and f- vector $(N_0(T), N_1(T), N_2(T))$, where $N_i(T) \in \mathbb{N}$ is the number of i-dimensional sub-simplices σ^i of T. Given a simplex σ we denote by $st(\sigma)$, (the star of σ), the union of all simplices of which σ is a face, and by $lk(\sigma)$, (the link of σ), is the union of all faces σ^f of the simplices in $st(\sigma)$ such that $\sigma^f \cap \sigma = \emptyset$. A Regge triangulation of a 2 -dimensional PL manifold M, (without boundary), is a homeomorphism $|T_l| \rightarrow M$ where each face of T is geometrically realized by a rectilinear simplex of variable edge-lengths $l(\sigma^1(k))$ (figure 1). A dynamical triangulation $|T_{l=a}| \rightarrow M$ is a particular case of a Regge PL-manifold realized

by rectilinear and equilateral simplices of edge-length $l(\sigma^1(k)) = a$.

The metric structure of a Regge triangulation is locally Euclidean everywhere except at the vertices σ^0, (the *bones*), where the sum of the dihedral angles, $\theta(\sigma^2)$, of the incident triangles σ^2's is in excess (negative curvature) or in defect (positive curvature) with respect to the 2π flatness constraint. The corresponding deficit angle ε is defined by $\varepsilon = 2\pi - \sum_{\sigma^2} \theta(\sigma^2)$, where the summation is extended to all 2 -dimensional simplices incident on the given bone σ^0. If K_T^0 denotes the (0)-skeleton of $|T_l| \to M$, (*i.e.*, the collection of vertices of the triangulation), then $M \backslash K_T^0$ is a flat Riemannian manifold, and any point in the interior of an r- simplex σ^r has a neighborhood homeomorphic to $B^r \times C(lk(\sigma^r))$, where B^r denotes the ball in $\mathbb{R}^n$ and $C(lk(\sigma^r))$ is the cone over the link $lk(\sigma^r)$, (the product $lk(\sigma^r) \times [0,1]$ with $lk(\sigma^r) \times \{1\}$ identified to a point). In particular, let us denote by $C|lk(\sigma^0(k))|$ the cone over the link of the vertex $\sigma^0(k)$. On any such a disk $C|lk(\sigma^0(k))|$ we can introduce a locally uniformizing complex coordinate $\zeta(k) \in \mathbb{C}$ in terms of which we can explicitly write down a conformal conical metric locally characterizing the singular structure of $|T_l| \to M$, *viz.* ,

$$e^{2u} \left|\zeta(k) - \zeta_k(\sigma^0(k))\right|^{-2\left(\frac{\varepsilon(k)}{2\pi}\right)} \left|d\zeta(k)\right|^2 , \tag{1}$$

where $\varepsilon(k)$ is the corresponding deficit angle, and $u : B^2 \to \mathbb{R}$ is a continuous function (C^2 on $B^2 - \{\sigma^0(k)\}$) such that, for $\zeta(k) \to \zeta_k(\sigma^0(k))$, we have $\left|\zeta(k) - \zeta_k(\sigma^0(k))\right| \frac{\partial u}{\partial \zeta(k)}$, and $\left|\zeta(k) - \zeta_k(\sigma^0(k))\right| \frac{\partial u}{\partial \bar{\zeta}(k)}$ both $\to 0$, [3]. Up to the presence of e^{2u}, we immediately recognize in such an expression the metric $g_{\theta(k)}$ of a Euclidean cone of total angle $\theta(k) = 2\pi - \varepsilon(k)$. The factor e^{2u} allows to move within the conformal class of all metrics possessing the same singular structure of the triangulated surface $|T_l| \to M$. We can profitably shift between the PL and the function theoretic point of view by exploiting standard techniques of complex analysis, and making contact with moduli space theory.

2.1 Curvature assignments and divisors.

In the case of dynamical triangulations, the picture simplifies considerably since the deficit angles are generated by the numbers $\#\{\sigma^2(h) \perp \sigma^0(i)\}$ of triangles incident on the $N_0(T)$ vertices, the *curvature assignments*, $\{q(k)\}_{k=1}^{N_0(T)} \in \mathbb{N}^{N_0(T)}$,

$$q(i) = \frac{2\pi - \varepsilon(i)}{\arccos(1/2)}. \tag{2}$$

For a regular triangulation we have $q(k) \geq 3$, and since each triangle has three vertices σ^0, the set of integers $\{q(k)\}_{k=1}^{N_0(T)}$ is constrained by

$$\sum_{k}^{N_0} q(k) = 3N_2 = 6\left[1 - \frac{\chi(M)}{N_0(T)}\right] N_0(T), \tag{3}$$

where $\chi(M)$ denotes the Euler-Poincaré characteristic of the surface, and where $6\left[1 - \frac{\chi(M)}{N_0(T)}\right]$, ($\simeq 6$ for $N_0(T) >> 1$), is the average value of the curvature

assignments $\{q(k)\}_{k=1}^{N_0}$. More generally we shall consider semi-simplicial complexes for which the constraint $q(k) \geq 3$ is removed. Examples of such configurations are afforded by triangulations with pockets, where two triangles are incident on a vertex, or by triangulations where the star of a vertex may contain just one triangle. We shall refer to such extended configurations as generalized (Regge and dynamical) triangulations.

The singular structure of the metric defined by (1) can be naturally summarized in a formal linear combination of the points $\{\sigma^0(k)\}$ with coefficients given by the corresponding deficit angles (normalized to 2π), i.e. in the *real divisor* [3]

$$Div(T) \doteq \sum_{k=1}^{N_0(T)} \left(-\frac{\varepsilon(k)}{2\pi}\right)\sigma^0(k) = \sum_{k=1}^{N_0(T)} \left(\frac{\theta(k)}{2\pi} - 1\right)\sigma^0(k) \tag{4}$$

supported on the set of bones $\{\sigma^0(i)\}_{i=1}^{N_0(T)}$. Note that the degree of such a divisor, defined by

$$|Div(T)| \doteq \sum_{k=1}^{N_0(T)} \left(\frac{\theta(k)}{2\pi} - 1\right) = -\chi(M) \tag{5}$$

is, for dynamical triangulations, a rewriting of the combinatorial constraint (3). In such a sense, the pair $(|T_{l=a}| \to M, Div(T))$, or shortly, $(T, Div(T))$, encodes the datum of the triangulation $|T_{l=a}| \to M$ and of a corresponding set of curvature assignments $\{q(k)\}$ on the vertices $\{\sigma^0(i)\}_{i=1}^{N_0(T)}$. The real divisor $|Div(T)|$ characterizes the Euler class of the pair $(T, Div(T))$ and yields for a corresponding Gauss-Bonnet formula. Explicitly, the Euler number associated with $(T, Div(T))$ is defined, [3], by

$$e(T, Div(T)) \doteq \chi(M) + |Div(T)|. \tag{6}$$

and the Gauss-Bonnet formula reads [3]:

Lemma 1 ***(Gauss-Bonnet for triangulated surfaces)*** *Let $(T, Div(T))$ be a triangulated surface with divisor*

$$Div(T) \doteq \sum_{k=1}^{N_0(T)} \left(\frac{\theta(k)}{2\pi} - 1\right)\sigma^0(k), \tag{7}$$

associated with the vertices $\{\sigma^0(k)\}_{k=1}^{N_0(T)}$. Let ds^2 be the conformal metric (1) representing the divisor $Div(T)$. Then

$$\frac{1}{2\pi}\int_M K dA = e(T, Div(T)), \tag{8}$$

where K and dA respectively are the curvature and the area element corresponding to the local metric $ds^2_{(k)}$.

Note that such a theorem holds for any singular Riemann surface Σ described by a divisor $Div(\Sigma)$ and not just for triangulated surfaces [3]. Since for a Regge (dynamical) triangulation, we have $e(T_a, Div(T)) = 0$, the Gauss-Bonnet formula implies

$$\frac{1}{2\pi}\int_M K dA = 0. \tag{9}$$

Thus, a triangulation $|T_l| \to M$ naturally carries a conformally flat structure. Clearly this is a rather obvious result, (since the metric in $M - \{\sigma^0(i)\}_{i=1}^{N_0(T)}$ is flat). However, it admits a not-trivial converse (recently proved by M. Troyanov, but, in a sense, going back to E. Picard) [3], [4]:

Theorem 2 *(**Troyanov-Picard**) Let $((M, \mathcal{C}_{sg}), Div)$ be a singular Riemann surface with a divisor such that $\mathrm{e}(M, Div) = 0$. Then there exists on M a unique (up to homothety) conformally flat metric representing the divisor Div.*

2.2 Conical Regge polytopes.

Let us consider the (first) barycentric subdivision of T. The closed stars, in such a subdivision, of the vertices of the original triangulation T_l form a collection of 2-cells $\{\rho^2(i)\}_{i=1}^{N_0(T)}$ characterizing the *conical* Regge polytope $|P_{T_l}| \to M$ (and its rigid equilateral specialization $|P_{T_a}| \to M$) barycentrically dual to $|T_l| \to M$. If $(\lambda(k), \chi(k))$ denote polar coordinates (based at $\sigma^0(k)$) of a point $p \in \rho^2(k)$, then $\rho^2(k)$ is geometrically realized as the space

$$\{(\lambda(k), \chi(k)) \ : \lambda(k) \geq 0; \chi(k) \in \mathbb{R}/(2\pi - \varepsilon(k))\mathbb{Z}\}/\,(0, \chi(k)) \sim (0, \chi'(k)) \tag{10}$$

endowed with the metric

$$d\lambda(k)^2 + \lambda(k)^2 d\chi(k)^2. \tag{11}$$

In other words, here we are not considering a rectilinear presentation of the dual cell complex P (where the PL-polytope is realized by flat polygonal 2-cells $\{\rho^2(i)\}_{i=1}^{N_0(T)}$) but rather a geometrical presentation $|P_{T_l}| \to M$ of P where the 2 -cells $\{\rho^2(i)\}_{i=1}^{N_0(T)}$ retain the conical geometry induced on the barycentric subdivision by the original metric (1) structure of $|T_l| \to M$ (figure 3).

2.3 Degenerations: hyperbolic cusps and cylindrical ends.

It is important to stress that whereas a Regge triangulation characterizes a unique (up to automorphisms) singular Euclidean structure, this latter actually allows for a more general type of metric triangulation. The point is that some of the vertices associated with a singular Euclidean structure can be characterized by deficit angles $\varepsilon(k) \to 2\pi i.e.$, $\sum_{\sigma^2(k)} \theta(\sigma^2(k)) = 0$. Such a situation corresponds to having the cone $C|lk(\sigma^0(k))|$ over the link $lk(\sigma^0(k))$ realized by a Euclidean cone of angle 0. This is a natural limiting case in a Regge triangulation, (think of a vertex where many long and thin triangles are incident), and it

is usually discarded as an unwanted pathology. However, there is really nothing pathological about that, since the corresponding 2-cell $\rho^2(k) \in |P_{T_l}| \to M$ can be naturally endowed with the conformal Euclidean structure obtained from (1) by setting $\frac{\varepsilon(k)}{2\pi} = 1$, *i.e.*

$$e^{2u} \left|\zeta(k) - \zeta_k(\sigma^0(k))\right|^{-2} |d\zeta(k)|^2 , \tag{12}$$

which (up to the conformal factor e^{2u}) is the flat metric on the half-infinite cylinder $\mathbb{S}^1 \times \mathbb{R}^+$ (a cylindrical end). Alternatively, one may consider $\rho^2(k)$ endowed with the geometry of a hyperbolic cusp (figure 4), *i.e.*, that of a half-infinite cylinder $\mathbb{S}^1 \times \mathbb{R}^+$ equipped with the hyperbolic metric $\lambda(k)^{-2}(d\lambda(k)^2 + d\chi(k)^2)$. The triangles incident on $\sigma^0(k)$ are then realized as hyperbolic triangles with the vertex $\sigma^0(k)$ located at $\lambda(k) = \infty$ and corresponding angle $\theta_k = 0$[5]. Since the Poincaré metric on the punctured disk $\{\zeta(k) \in C|\ 0 < \left|\zeta(k) - \zeta_k(\sigma^0(k))\right| < 1\}$ is

$$\left(\left|\zeta(k) - \zeta_k(\sigma^0(k))\right| \ln \frac{1}{|\zeta(k) - \zeta_k(\sigma^0(k))|}\right)^{-2} |d\zeta(k)|^2 , \tag{13}$$

one can shift from the Euclidean to the hyperbolic metric by setting

$$e^{2u} = \left(\ln \frac{1}{|\zeta(k) - \zeta_k(\sigma^0(k))|}\right)^{-2} , \tag{14}$$

and the two points of view are strictly related.

At any rate the presence of hyperbolic cusps or cylindrical ends is consistent with a singular Euclidean structure as long as the associated divisor satisfies the topological constraint (5), which we can rewrite as

$$\sum_{\{\frac{\varepsilon(k)}{2\pi} \neq 1\}} \left(-\frac{\varepsilon(k)}{2\pi}\right) = 2g - 2 + \#\left\{\sigma^0(h)|\ \frac{\varepsilon(h)}{2\pi} = 1\right\}. \tag{15}$$

In particular, we can have the limiting case of the singular Euclidean structure associated with a genus g surface triangulated with $N_0 - 1$ hyperbolic vertices $\{\sigma^0(k)\}_{k=1}^{N_0-1}$ (or, equivalently, with $N_0 - 1$ cylindrical ends) and just one standard conical vertex, $\sigma^0(N_0)$, supporting the deficit angle

$$-\frac{\varepsilon(N_0)}{2\pi} = 2g - 2 + (N_0 - 1). \tag{16}$$

2.4 Conical sectors and moduli.

We can represent finer details of the geometry of $(U_{\rho^2(k)}, ds^2_{(k)})$ by opening the cone into its constituent conical sectors. To motivate such a representation, let $W_\alpha(k)$, $\alpha = 1, ..., q(k)$, be the barycenters of the edges $\sigma^1(\alpha) \in |T_l| \to M$ incident on $\sigma^0(k)$, and intercepting the boundary $\partial(\rho^2(k))$ of the polygonal cell $\rho^2(k)$. Denote by $l(\partial(\rho^2(k)))$ the length of $\partial(\rho^2(k))$, and by $\widehat{L}_\alpha(k)$ the length of the polygonal $\partial(\rho^2(k))$ between the points $W_\alpha(k)$ and $W_{\alpha+1}(k)$ (with

α defined $\bmod q(k)$). In the uniformization $\zeta(k)$ of $C|lk(\sigma^0(k))|$, the points $\{W_\alpha(k)\}$ characterize a corresponding set of points on the circumference $\{\zeta(k) \in \mathbb{C} \mid |\zeta(k)| = l(\partial(\rho^2(k)))\}$, (for simplicity, we have set $\zeta_k(\sigma^0(k)) = 0$), and an associated set of $q(k)$ generators $\{\overline{W_\alpha(k)\sigma^0(k)}\}$ on the cone $(U_{\rho^2(k)}, ds^2_{(k)})$. Such generators mark $q(k)$ conical sectors

$$S_\alpha(k) \doteq \left(c_\alpha(k), \frac{l(\partial(\rho^2(k)))}{\theta(k)}, \vartheta_\alpha(k)\right), \tag{17}$$

with base

$$c_\alpha(k) \doteq \left\{|\zeta(k)| = l(\partial(\rho^2(k))), \arg W_\alpha(k) \leq \arg \zeta(k) \leq \arg W_{\alpha+1}(k)\right\}, \tag{18}$$

slant radius $\frac{l(\partial(\rho^2(k)))}{\theta(k)}$, and with angular opening

$$\vartheta_\alpha(k) \doteq \frac{\widehat{L}_\alpha(k)}{l(\partial(\rho^2(k)))}\theta(k), \tag{19}$$

where $\theta(k) = 2\pi - \varepsilon(k)$ is the given conical angle. Since $\sum_{\alpha=1}^{q(k)} \vartheta_\alpha(k) = \theta(k)$, the $\{\vartheta_\alpha(k)\}$ are the representatives, in the uniformization $(U_{\rho^2(k)}, ds^2_{(k)})$, of the $q(k)$ vertex angles generating the deficit angle $\varepsilon(k)$ of $C|lk(\sigma^0(k))|$. In particular, we can formally represent the cone $(U_{\rho^2(k)}, ds^2_{(k)})$ as

$$(U_{\rho^2(k)}, ds^2_{(k)}) = \cup_{\alpha=1}^{q(k)} S_\alpha(k). \tag{20}$$

If we split open the vertex of the cone and of the associated conical sectors $S_\alpha(k)$, then the conical geometry of $(U_{\rho^2(k)}, ds^2_{(k)})$ can be equivalently described by a cylindrical strip of height $\frac{l(\partial(\rho^2(k)))}{\theta(k)}$ decorating the boundary of $\rho^2(k)$. Each sector $S_\alpha(k)$ in the cone gives rise, in such a picture, to a rectangular region in the cylindrical strip (figure 5). It is profitable to explicitly represent any such a region, in the complex plane of the variable $z = x + \sqrt{-1}y$, upside down according to

$$R_{\vartheta_\alpha(k)}(k) \doteq \left\{z \in \mathbb{C}|\ 0 \leq x \leq \frac{l(\partial(\rho^2(k)))}{\theta(k)}, 0 \leq y \leq \widehat{L}_\alpha(k)\right\}. \tag{21}$$

We can go a step further, and by means of the conformal transformation

$$W_\alpha(k) = \exp\left[\frac{2\pi\sqrt{-1}\theta(k)}{l(\partial(\rho^2(k)))}z\right], \tag{22}$$

we can map the rectangle $R_{\vartheta_\alpha(k)}(k)$ into the annulus (figure 6)

$$\Delta_{\vartheta_\alpha(k)}(k) \doteq \{W(k) \in \mathbb{C}|\ |t_\alpha(k)| < |W(k)| < 1\}, \tag{23}$$

where

$$|t_\alpha(k)| \doteq \exp\left[-2\pi\vartheta_\alpha(k)\right]. \tag{24}$$

Note that

$$\frac{1}{2\pi}\ln\left(\frac{1}{|t_\alpha(k)|}\right)=\vartheta_\alpha(k), \tag{25}$$

is the modulus of $\Delta_{\vartheta_\alpha(k)}(k)$.

Such a remark motivates the analysis of the geometry of (random) Regge triangulations from the point of view of moduli theory. In this connection note that for a given set of perimeters $\{l(\partial(\rho^2(k)))\}_{k=1}^{N_0}$ and deficit angles $\{\varepsilon(k)\}_{k=1}^{N_0}$ there are $(N_1(T)-N_0(T))$ free angles $\vartheta_\alpha(k)$. This follows by observing that, for given $l(\partial(\rho^2(k)))$ and $\varepsilon(k)$, the angles $\vartheta_\alpha(k)$ are characterized (see (19)) by the $\widehat{L}_\alpha(k)$. These latter are in a natural correspondence with the N_1 edges of $|P_{T_l}|\rightarrow M$, and among them we have the N_0 constraints $\sum_\alpha^{q(k)}\widehat{L}_\alpha(k)=l(\partial(\rho^2(k)))$. From $N_0(T)-N_1(T)+N_2(T)=2-2g$, and the relation $2N_1(T)=3N_2(T)$ associated with the trivalency, we get $N_1(T)-N_0(T)=2N_0(T)+6g-6$, which exactly corresponds to the real dimension of the moduli space $\mathfrak{M}_{g,N_0}$ of genus g Riemann surfaces $((M;N_0),\mathcal{C})$ with N_0 punctures, (see below).

3 Ribbon graphs on Regge Polytopes

The geometrical realization of the 1-skeleton of the conical Regge polytope $|P_{T_l}|\rightarrow M$ is a 3-valent graph

$$\Gamma=(\{\rho^0(k)\},\{\rho^1(j)\}) \tag{26}$$

where the vertex set $\{\rho^0(k)\}_{k=1}^{N_2(T)}$ is identified with the barycenters of the triangles $\{\sigma^o(k)\}_{k=1}^{N_2(T)}\in|T_l|\rightarrow M$, whereas each edge $\rho^1(j)\in\{\rho^1(j)\}_{j=1}^{N_1(T)}$ is generated by two half-edges $\rho^1(j)^+$ and $\rho^1(j)^-$ joined through the barycenters $\{W(h)\}_{h=1}^{N_1(T)}$ of the edges $\{\sigma^1(h)\}$ belonging to the original triangulation $|T_l|\rightarrow M$. If we formally introduce a ghost-vertex of a degree 2 at each middle point $\{W(h)\}_{h=1}^{N_1(T)}$, then the actual graph naturally associated to the 1-skeleton of $|P_{T_l}|\rightarrow M$ is the edge-refinement [2] of $\Gamma=(\{\rho^0(k)\},\{\rho^1(j)\})$, *i.e.*

$$\Gamma_{ref}=\left(\{\rho^0(k)\}\bigsqcup_{h=1}^{N_1(T)}\{W(h)\},\{\rho^1(j)^+\}\bigsqcup_{j=1}^{N_1(T)}\{\rho^1(j)^-\}\right), \tag{27}$$

as it can be seen in figure 7.

The natural automorphism group $Aut(P_l)$ of $|P_{T_l}|\rightarrow M$, (*i.e.*, the set of bijective maps $\Gamma=(\{\rho^0(k)\},\{\rho^1(j)\})\rightarrow\widetilde{\Gamma}=(\{\widetilde{\rho^0(k)}\},\{\widetilde{\rho^1(j)}\}$ preserving the incidence relations defining the graph structure), is the automorphism group of its edge refinement [2], $Aut(P_l)\doteq Aut(\Gamma_{ref})$. The locally uniformizing complex coordinate $\zeta(k)\in\mathbb{C}$ in terms of which we can explicitly write down the singular Euclidean metric (1) around each vertex $\sigma^0(k)\in|T_l|\rightarrow M$, provides a (counterclockwise) orientation in the 2-cells of $|P_{T_l}|\rightarrow M$. Such an orientation gives rise to a cyclic ordering on the set of half-edges $\{\rho^1(j)^\pm\}_{j=1}^{N_1(T)}$ incident

on the vertices $\{\rho^0(k)\}_{k=1}^{N_2(T)}$. According to these remarks, the 1-skeleton of $|P_{T_l}| \rightarrow M$ is a ribbon (or fat) graph [6], a graph Γ together with a cyclic ordering on the set of half-edges incident to each vertex of Γ. Conversely, any ribbon graph Γ characterizes an oriented surface $M(\Gamma)$ with boundary possessing Γ as a spine, (*i.e.*, the inclusion $\Gamma \hookrightarrow M(\Gamma)$ is a homotopy equivalence). In this way (the edge-refinement of) the 1-skeleton of a generalized conical Regge polytope $|P_{T_l}| \rightarrow M$ is in a one-to-one correspondence with trivalent metric ribbon graphs. The set of all such trivalent ribbon graph Γ with given edge-set $e(\Gamma)$ can be characterized [2], [7] as a space homeomorphic to $\mathbb{R}_+^{|e(\Gamma)|}$, ($|e(\Gamma)|$ denoting the number of edges in $e(\Gamma)$), topologized by the standard ϵ-neighborhoods $U_\epsilon \subset \mathbb{R}_+^{|e(\Gamma)|}$. The automorphism group $Aut(\Gamma)$ acts naturally on such a space via the homomorphism $Aut(\Gamma) \rightarrow \mathfrak{G}_{e(\Gamma)}$, where $\mathfrak{G}_{e(\Gamma)}$ denotes the symmetric group over $|e(\Gamma)|$ elements, and the resulting quotient space $\mathbb{R}_+^{|e(\Gamma)|}/Aut(\Gamma)$ is a differentiable orbifold.

3.1 The space of 1-skeletons of Regge polytopes.

Let $Aut_\partial(P_l) \subset Aut(P_l)$, denote the subgroup of ribbon graph automorphisms of the (trivalent) 1-skeleton Γ of $|P_{T_l}| \rightarrow M$ that preserve the (labeling of the) boundary components of Γ. Then, the space K_1RP_{g,N_0}^{met} of 1 -skeletons of conical Regge polytopes $|P_{T_l}| \rightarrow M$, with $N_0(T)$ labelled boundary components, on a surface M of genus g can be defined by [2]

$$K_1RP_{g,N_0}^{met} = \bigsqcup_{\Gamma \in RGB_{g,N_0}} \frac{\mathbb{R}_+^{|e(\Gamma)|}}{Aut_\partial(P_l)}, \tag{28}$$

where the disjoint union is over the subset of all trivalent ribbon graphs (with labelled boundaries) satisfying the topological stability condition $2-2g-N_0(T) < 0$, and which are dual to generalized triangulations. It follows, (see [2] theorems 3.3, 3.4, and 3.5), that the set K_1RP_{g,N_0}^{met} is locally modelled on a stratified space constructed from the components (rational orbicells) $\mathbb{R}_+^{|e(\Gamma)|}/Aut_\partial(P_l)$ by means of a (Whitehead) expansion and collapse procedure for ribbon graphs, which amounts to collapsing edges and coalescing vertices, (the Whitehead move in $|P_{T_l}| \rightarrow M$ is the dual of the familiar flip move [6] for triangulations). Explicitly, if $l(t) = tl$ is the length of an edge $\rho^1(j)$ of a ribbon graph $\Gamma_{l(t)} \in K_1RP_{g,N_0}^{met}$, then, as $t \rightarrow 0$, we get the metric ribbon graph $\widehat{\Gamma}$ which is obtained from $\Gamma_{l(t)}$ by collapsing the edge $\rho^1(j)$. By exploiting such construction, we can extend the space K_1RP_{g,N_0}^{met} to a suitable closure $\overline{K_1RP}_{g,N_0}^{met}$ [7], (this natural topology on K_1RP_{g,N_0}^{met} shows that, at least in two-dimensional quantum gravity, the set of Regge triangulations with *fixed connectivity* does not explore the full configurational space of the theory). The open cells of K_1RP_{g,N_0}^{met}, being associated with trivalent graphs, have dimension provided by the number $N_1(T)$ of edges of $|P_{T_l}| \rightarrow M$, *i.e.*

$$\dim \left[K_1RP_{g,N_0}^{met}\right] = N_1(T) = 3N_0(T) + 6g - 6. \tag{29}$$

There is a natural projection

$$p : K_1 RP^{met}_{g,N_0} \longrightarrow \mathbb{R}_+^{N_0(T)} \tag{30}$$
$$\Gamma \longmapsto p(\Gamma) = (l_1, ..., l_{N_0(T)}),$$

where $(l_1, ..., l_{N_0(T)})$ denote the perimeters of the polygonal 2-cells $\{\rho^2(j)\}$ of $|P_{T_l}| \rightarrow M$. With respect to the topology on the space of metric ribbon graphs, the orbifold $K_1 RP^{met}_{g,N_0}$ endowed with such a projection acquires the structure of a cellular bundle. For a given sequence $\{l(\partial(\rho^2(k)))\}$, the fiber

$$p^{-1}(\{l(\partial(\rho^2(k)))\}) = \left\{|P_{T_l}| \rightarrow M \in K_1 RP^{met}_{g,N_0} : \{l_k\} = \{l(\partial(\rho^2(k)))\}\right\} \tag{31}$$

is the set of all generalized conical Regge polytopes with the given set of perimeters. If we take into account the $N_0(T)$ constraints associated with the perimeters assignments, it follows that the fibers $p^{-1}(\{l(\partial(\rho^2(k)))\})$ have dimension provided by

$$\dim\left[p^{-1}(\{l(\partial(\rho^2(k)))\}\right] = 2N_0(T) + 6g - 6, \tag{32}$$

which again corresponds to the real dimension of the moduli space $\mathfrak{M}_{g,N_0}$ of N_0-pointed Riemann surfaces of genus g.

3.2 Orbifold labelling and dynamical triangulations.

Let us denote by

$$\Omega_{T_a} \doteq \frac{\mathbb{R}_+^{|e(\Gamma)|}}{Aut_\partial(P_{T_a})} \tag{33}$$

the rational cell associated with the 1-skeleton of the conical polytope $|P_{T_a}| \rightarrow M$ dual to a dynamical triangulation $|T_{l=a}| \rightarrow M$. The orbicell (33) contains the ribbon graph associated with $|P_{T_a}| \rightarrow M$ and all (trivalent) metric ribbon graphs $|P_{T_L}| \rightarrow M$ with the same combinatorial structure of $|P_{T_a}| \rightarrow M$ but with all possible length assignments $\{l(\rho^1(h))\}_{h=1}^{N_1(T)}$ associated with the corresponding set of edges $\{\rho^1(h)\}_1^{N_1(T)}$. The orbicell Ω_{T_a} is naturally identified with the convex polytope (of dimension $(2N_0(T) + 6g - 6)$) in $\mathbb{R}_+^{N_1(T)}$ defined by

$$\left\{\{l(\rho^1(j))\} \in \mathbb{R}_+^{N_1(T)} : \sum_{j=1}^{q(k)} A^j_{(k)}(T_a) l(\rho^1(j)) \ = \frac{\sqrt{3}}{3} a q(k), \ k = 1, ..., N_0 \right\}, \tag{34}$$

where $A^j_{(k)}(T_a)$ is a $(0,1)$ indicator matrix, depending on the given dynamical triangulation $|T_{l=a}| \rightarrow M$, with $A^j_{(k)}(T_a) = 1$ if the edge $\rho^1(j)$ belongs to $\partial(\rho^2(k))$, and 0 otherwise, and $\frac{\sqrt{3}}{3} a q(k)$ is the perimeter length $l(\partial(\rho^2(k)))$ in terms of the corresponding curvature assignment $q(k)$. Note that $|P_{T_a}| \rightarrow M$ appears as the barycenter of such a polytope.

Since the cell decomposition (28) of the space of trivalent metric ribbon graphs $K_1 RP^{met}_{g,N_0}$ depends only on the combinatorial type of the ribbon graph,

we can use the equilateral polytopes $|P_{T_a}| \to M$, dual to dynamical triangulations, as the set over which the disjoint union in (28) runs. Thus we can write

$$K_1 RP^{met}_{g,N_0} = \bigsqcup_{\mathcal{DT}(N_0)} \Omega_{T_a}, \tag{35}$$

where

$$\mathcal{DT}_g(N_0) \doteq \left\{ |T_{l=a}| \to M \ : (\sigma^0(k)) \ k = 1, ..., N_0(T) \right\} \tag{36}$$

denote the set of distinct generalized dynamically triangulated surfaces of genus g, with a given set of $N_0(T)$ ordered labelled vertices.

Note that, even if the set $\mathcal{DT}_g(N_0)$ can be considered (through barycentrical dualization) a well-defined subset of $K_1RP^{met}_{g,N_0}$, it is not an orbifold over $\mathbb{N}$ [8]. For this latter reason, the analysis of the metric stuctures over (generalized) polytopes requires the use of the full orbicells Ω_{T_a} and we cannot limit our discussion to equilateral polytopes.

3.3 The ribbon graph parametrization of the moduli space

We start by recalling that the moduli space $\mathfrak{M}_{g,N_0}$ of genus g Riemann surfaces with N_0 punctures is a dense open subset of a natural compactification (Knudsen-Deligne-Mumford) in a connected, compact complex orbifold denoted by $\overline{\mathfrak{M}}_{g,N_0}$. This latter is, by definition, the moduli space of stable N_0-pointed curves of genus g, where a stable curve is a compact Riemann surface with at most ordinary double points such that all of its parts are hyperbolic. The closure $\partial\mathfrak{M}_{g,N_0}$ of $\mathfrak{M}_{g,N_0}$ in $\overline{\mathfrak{M}}_{g,N_0}$ consists of stable curves with double points, and gives rise to a stratification decomposing $\overline{\mathfrak{M}}_{g,N_0}$ into subvarieties. By definition, a stratum of codimension k is the component of $\overline{\mathfrak{M}}_{g,N_0}$ parametrizing stable curves (of fixed topological type) with k double points.

The complex analytic geometry of the space of conical Regge polytopes which we will discuss in the next section generalizes the well-known bijection (a homeomorphism of orbifolds) between the space of metric ribbon graphs $K_1RP^{met}_{g,N_0}$ (which forgets the conical geometry) and the moduli space $\mathfrak{M}_{g,N_0}$ of genus g Riemann surfaces $((M; N_0), \mathcal{C})$ with $N_0(T)$ punctures [2], [7]. This bijection results in a local parametrization of $\mathfrak{M}_{g,N_0}$ defined by

$$\begin{aligned} h : K_1RP^{met}_{g,N_0} &\to \mathfrak{M}_{g,N_0} \times R^N_+ \\ \Gamma &\longmapsto [((M; N_0), \mathcal{C}), l_i] \end{aligned} \tag{37}$$

where $(l_1, ..., l_{N_0})$ is an ordered n-tuple of positive real numbers and Γ is a metric ribbon graphs with $N_0(T)$ labelled boundary lengths $\{l_i\}$ (figure 8).

If $\overline{K_1RP}^{met}_{g,N_0}$ is the closure of $K_1RP^{met}_{g,N_0}$, then the bijection h extends to $\overline{K_1RP}^{met}_{g,N_0} \to \overline{\mathfrak{M}}_{g,N_0} \times R^{N_0}_+$ in such a way that a ribbon graph $\Gamma \in \overline{RGP}^{met}_{g,N_0}$ is mapped in two (stable) surfaces M_1 and M_2 with $N_0(T)$ punctures if and only if there exists an homeomorphism between M_1 and M_2 preserving the (labelling of the) punctures, and is holomorphic on each irreducible component containing one of the punctures.

According to Kontsevich [9], corresponding to each marked polygonal 2-cells $\{\rho^2(k)\}$ of $|P_{T_l}| \to M$ there is a further (combinatorial) bundle map

$$\mathcal{CL}_k \to K_1 RP^{met}_{g,N_0} \tag{38}$$

whose fiber over $(\Gamma, \rho^2(1), ..., \rho^2(N_0))$ is provided by the boundary cycle $\partial\rho^2(k)$, (recall that each boundary $\partial\rho^2(k)$ comes with a positive orientation). To any such cycle one associates [7], [9] the corresponding perimeter map $l(\partial(\rho^2(k))) = \sum l(\rho^1(h_\alpha))$ which then appears as defining a natural connection on $\mathcal{CL}_k$. The piecewise smooth 2-form defining the curvature of such a connection,

$$\omega_k(\Gamma) = \sum_{1 \le h_\alpha < h_\beta \le q(k)-1} d\left(\frac{l(\rho^1(h_\alpha))}{l(\partial\rho^2(k))}\right) \wedge d\left(\frac{l(\rho^1(h_\beta))}{l(\partial\rho^2(k))}\right), \tag{39}$$

is invariant under rescaling and cyclic permutations of the $l(\rho^1(h_\mu))$, and is a combinatorial representative of the Chern class of the line bundle $\mathcal{CL}_k$.

It is important to stress that even if ribbon graphs can be thought of as arising from Regge polytopes (with variable connectivity), the morphism (37) only involves the ribbon graph structure and the theory can be (and actually is) developed with no reference at all to a particular underlying triangulation. In such a connection, the role of dynamical triangulations has been slightly overemphasized, they simply provide a convenient way of labelling the different combinatorial strata of the mapping (37), but, by themselves they do not define a combinatorial parametrization of $\overline{\mathfrak{M}}_{g,N_0}$ for any finite N_0. However, it is very useful, at least for the purposes of quantum gravity, to remember the possible genesis of a ribbon graph from an underlying triangulation and be able to exploit the further information coming from the associated conical geometry. Such an information cannot be recovered from the ribbon graph itself (with the notable exception of equilateral ribbon graphs, which can be associated with dynamical triangulations), and must be suitably codified by adding to the boundary lengths $\{l_i\}$ of the graph a further decoration. This can be easily done by explicitly connecting Regge polytopes to punctured Riemann surfaces.

4 Regge polytopes and punctured Riemann surfaces.

As suggested by (1), the polyhedral metric associated with the vertices $\{\sigma^0(i)\}$ of a (generalized) Regge triangulation $|T_l| \to M$, can be conveniently described in terms of complex function theory. We can extend the ribbon graph uniformization of [2] and associate with $|P_{T_l}| \to M$ a complex structure $((M; N_0), \mathcal{C})$ (a punctured Riemann surface) which is, in a well-defined sense, dual to the structure (1) generated by $|T_l| \to M$. Let $\rho^2(k)$ be the generic two-cell $\in |P_{T_l}| \to M$ barycentrically dual to the vertex $\sigma^0(k) \in |T_l| \to M$. To the generic edge $\rho^1(h)$

of $\rho^2(k)$ we associate a complex uniformizing coordinate $z(h)$ defined in the strip

$$U_{\rho^1(h)} \doteq \{z(h) \in \mathbb{C} \mid 0 < \operatorname{Re} z(h) < l(\rho^1(h))\}, \tag{40}$$

$l(\rho^1(h))$ being the length of the edge considered. The uniformizing coordinate $w(j)$, corresponding to the generic 3-valent vertex $\rho^0(j) \in \rho^2(k)$, is defined in the open set

$$U_{\rho^0(j)} \doteq \{w(j) \in \mathbb{C} \mid |w(j)| < \delta,\ w(j)[\rho^0(j)] = 0\}, \tag{41}$$

where $\delta > 0$ is a suitably small constant. Finally, the two-cell $\rho^2(k)$ is uniformized in the unit disk

$$U_{\rho^2(k)} \doteq \{\zeta(k) \in \mathbb{C} \mid |\zeta(k)| < 1,\ \zeta(k)[\sigma^0(k)] = 0\}, \tag{42}$$

where $\sigma^0(k)$ is the vertex $\in |T_l| \to M$ corresponding to the given two-cell (figure 9).

The various uniformizations $\{w(j), U_{\rho^0(j)}\}_{j=1}^{N_2(T)}$, $\{z(h), U_{\rho^1(h)}\}_{h=1}^{N_1(T)}$, and $\{\zeta(k), U_{\rho^2(k)}\}_{k=1}^{N_0(T)}$ can be coherently glued together by noting that to each edge $\rho^1(h) \in \rho^2(k)$ we can associate the standard quadratic differential on $U_{\rho^1(h)}$ given by

$$\phi(h)|_{\rho^1(h)} = dz(h) \otimes dz(h). \tag{43}$$

Such $\phi(h)|_{\rho^1(h)}$ can be extended to the remaining local uniformizations $U_{\rho^0(j)}$, and $U_{\rho^2(k)}$, by exploiting a classic result in Riemann surface theory according to which a quadratic differential ϕ has a finite number of zeros $n_{zeros}(\phi)$ with orders k_i and a finite number of poles $n_{poles}(\phi)$ of order s_i such that

$$\sum_{i=1}^{n_{zero}(\phi)} k_i - \sum_{i=1}^{n_{pole}(\phi)} s_i = 4g - 4. \tag{44}$$

In our case we must have $n_{zeros}(\phi) = N_2(T)$ with $k_i = 1$, (corresponding to the fact that the 1-skeleton of $|P_l| \to M$ is a trivalent graph), and $n_{poles}(\phi) = N_0(T)$ with $s_i = s\ \forall i$, for a suitable positive integer s. According to such remarks (44) reduces to

$$N_2(T) - sN_0(T) = 4g - 4. \tag{45}$$

¿From the Euler relation $N_0(T) - N_1(T) + N_2(T) = 2 - 2g$, and $2N_1(T) = 3N_2(T)$ we get $N_2(T) - 2N_0(T) = 4g - 4$. This is consistent with (45) if and only if $s = 2$. Thus the extension ϕ of $\phi(h)|_{\rho^1(h)}$ along the 1-skeleton of $|P_l| \to M$ must have $N_2(T)$ zeros of order 1 corresponding to the trivalent vertices $\{\rho^0(j)\}$ of $|P_l| \to M$ and $N_0(T)$ quadratic poles corresponding to the polygonal cells $\{\rho^2(k)\}$ of perimeter lengths $\{l(\partial(\rho^2(k)))\}$. Around a zero of order one and a pole of order two, every (Jenkins-Strebel [10]) quadratic differential ϕ has a canonical local structure which (along with (43)) is given by [2][10]

$$(|P_{T_l}| \to M) \to \phi \doteq \begin{cases} \phi(h)|_{\rho^1(h)} = dz(h) \otimes dz(h), \\ \phi(j)|_{\rho^0(j)} = \frac{9}{4} w(j) dw(j) \otimes dw(j), \\ \phi(k)|_{\rho^2(k)} = -\frac{\left[l(\partial(\rho^2(k)))\right]^2}{4\pi^2 \zeta^2(k)} d\zeta(k) \otimes d\zeta(k), \end{cases} \tag{46}$$

where $\{\rho^0(j), \rho^1(h), \rho^2(k)\}$ runs over the set of vertices, edges, and 2-cells of $|P_{T_l}| \to M$. Since $\phi(h)|_{\rho^1(h)}$, $\phi(j)|_{\rho^0(j)}$, and $\phi(k)|_{\rho^2(k)}$ must be identified on the non-empty pairwise intersections $U_{\rho^0(j)} \cap U_{\rho^1(h)}$, $U_{\rho^1(h)} \cap U_{\rho^2(k)}$ we can associate to the polytope $|P_{T_l}| \to M$ a complex structure $((M; N_0), \mathcal{C})$ by coherently gluing, along the pattern associated with the ribbon graph Γ, the local uniformizations $\{U_{\rho^0(j)}\}_{j=1}^{N_2(T)}$, $\{U_{\rho^1(h)}\}_{h=1}^{N_1(T)}$, and $\{U_{\rho^2(k)}\}_{k=1}^{N_0(T)}$. Explicitly, let $\{U_{\rho^1(j_\alpha)}\}$, $\alpha = 1, 2, 3$ be the three generic open strips associated with the three cyclically oriented edges $\{\rho^1(j_\alpha)\}$ incident on the generic vertex $\rho^0(j)$. Then the uniformizing coordinates $\{z(j_\alpha)\}$ are related to $w(j)$ by the transition functions

$$w(j) = e^{2\pi i \frac{\alpha-1}{3}} z(j_\alpha)^{\frac{2}{3}}, \qquad \alpha = 1, 2, 3. \tag{47}$$

Note that in such uniformization the vertices $\{\rho^0(j)\}$ do not support conical singularities since each strip $U_{\rho^1(j_\alpha)}$ is mapped by (47) into a wedge of angular opening $\frac{2\pi}{3}$. This is consistent with the definition of $|P_{T_l}| \to M$ according to which the vertices $\{\rho^0(j)\} \in |P_{T_l}| \to M$ are the barycenters of the flat $\{\sigma^2(j)\} \in |T_l| \to M$. Similarly, if $\{U_{\rho^1(k_\beta)}\}$, $\beta = 1, 2, ..., q(k)$ are the open strips associated with the $q(k)$ (oriented) edges $\{\rho^1(k_\beta)\}$ boundary of the generic polygonal cell $\rho^2(k)$, then the transition functions between the corresponding uniformizing coordinate $\zeta(k)$ and the $\{z(k_\beta)\}$ are given by [2]

$$\zeta(k) = \exp\left(\frac{2\pi i}{l(\partial(\rho^2(k)))}\left(\sum_{\beta=1}^{\nu-1} l(\rho^1(k_\beta)) + z(k_\nu)\right)\right), \qquad \nu = 1, ..., q(k), \tag{48}$$

with $\sum_{\beta=1}^{\nu-1} \cdot \doteq 0$, for $\nu = 1$.

4.1 A Parametrization of the conical geometry.

Note that for any closed curve $c : \mathbb{S}^1 \to U_{\rho^2(k)}$, homotopic to the boundary of $\overline{U}_{\rho^2(k)}$, we get

$$\oint_c \sqrt{\phi(k)_{\rho^2(k)}} = l(\partial(\rho^2(k))). \tag{49}$$

which shows that the geometry associated with $\phi(k)_{\rho^2(k)}$ is described by the cylindrical metric canonically associated with a quadratic differential with a second order pole,*i.e.*

$$|\phi(k)_{\rho^2(k)}| = \frac{\left[l(\partial(\rho^2(k)))\right]^2}{4\pi^2|\zeta(k)|^2}|d\zeta(k)|^2. \tag{50}$$

Thus, as already recalled (section 2.3) the punctured disk $\Delta_k^* \subset U_{\rho^2(k)}$

$$\Delta_k^* \doteq \{\zeta(k) \in \mathbb{C} \mid 0 < |\zeta(k)| < 1\}, \tag{51}$$

endowed with the flat metric $|\phi(k)_{\rho^2(k)}|$, is isometric to a flat semi-infinite cylinder. It is perhaps worthwhile stressing that even if this latter geometry is perfectly consistent with the metric ribbon graph structure, it is not really the

natural metric to use if we wish to explicitly keep track of the conical Regge polytope (and the associated triangulation) which generate the given ribbon graph. For instance, in the Kontsevich-Witten model, the ribbon graph structure and the associated geometry $(\Delta_k^*, |\phi(k)_{\rho^2(k)}|)$ of the cells $U_{\rho^2(k)}$ is sufficient to combinatorially describe intersection theory on moduli space. However, a full-fledged study of 2D simplicial quantum gravity (*e.g.*, of Liouville theory) requires a study of the full Regge geometry $(\Delta_k^*, ds^2_{(k)})$.

We can keep track of the conical geometry of the polygonal cell $\rho^2(k) \in |P_{T_l}| \to M$ by noticing that for a given deficit angle $\varepsilon(k) = 2\pi - \theta(k)$, the conical geometry (1) and the cylindrical geometry (50) can be conformally related on each punctured disk $\Delta_k^* \subset U_{\rho^2(k)}$ according to

$$ds^2_{(k)} \doteq \frac{\left[l(\partial(\rho^2(k)))\right]^2}{4\pi^2} |\zeta(k)|^{-2\left(\frac{\varepsilon(k)}{2\pi}\right)} |d\zeta(k)|^2 =$$

$$= \frac{\left[l(\partial(\rho^2(k)))\right]^2}{4\pi^2} |\zeta(k)|^{2\left(\frac{\theta(k)}{2\pi}\right)} \frac{|d\zeta(k)|^2}{|\zeta(k)|^2}. \tag{52}$$

It follows that we can apply the explicit construction [2] of the mapping (37) for defining the decorated Riemann surface corresponding to a conical Regge polytope (figure 10). Then, an obvious adaptation of theorem 4.2 of [2] provides

Proposition 3 *If $\{p_k\}_{k=1}^{N_0} \in M$ denotes the set of punctures corresponding to the decorated vertices $\{\sigma^0(k), \frac{\varepsilon(k)}{2\pi}\}_{k=1}^{N_0}$ of the triangulation $|T_l| \to M$, let RP^{met}_{g,N_0} be the space of conical Regge polytopes $|P_{T_l}| \to M$, with $N_0(T)$ labelled vertices, and let $\mathfrak{P}_{g,N_0}$ be the moduli space $\mathfrak{M}_{g,N_0}$ decorated with the local metric uniformizations $(\zeta(k), ds^2_{(k)})$ around each puncture p_k. Then the map*

$$\Upsilon : RP^{met}_{g,N_0} \longrightarrow \mathfrak{P}_{g,N_0}$$

provided by

$$\Upsilon : (|P_{T_l}| \to M) \longrightarrow ((M; N_0), \mathcal{C}); \{ds^2_{(k)}\}) = \tag{53}$$

$$= \bigcup_{\{\rho^0(j)\}}^{N_2(T)} U_{\rho^0(j)} \bigcup_{\{\rho^1(h)\}}^{N_1(T)} U_{\rho^1(h)} \bigcup_{\{\rho^2(k)\}}^{N_0(T)} (U_{\rho^2(k)}, ds^2_{(k)}),$$

defines the decorated, N_0-pointed, Riemann surface $((M; N_0), \mathcal{C})$ canonically associated with the conical Regge polytope $|P_{T_l}| \to M$.

In other words, through the morphism Υ defined by the gluing maps (47) and (48) one can generate, from a polytope barycentrically dual to a (generalized) Regge triangulation, a Riemann surface $((M; N_0), \mathcal{C}) \in \overline{\mathfrak{M}}_{g,N_0}$.

Such a surface naturally carries the decoration provided by a choice of local coordinates $\zeta(k)$ around each puncture, of the corresponding meromorphic quadratic differential $\phi(k)|_{\rho^2(k)}$ (figure 11) and of the associated conical metric $ds^2_{(k)}$. It is through such a decoration that the punctured Riemann surface $((M; N_0), \mathcal{C})$ keeps track of the metric geometry of the conical Regge polytope $|P_{T_l}| \to M$ out of which $((M; N_0), \mathcal{C})$ has been generated.

4.2 Opening of the cones and deformations of the Regge geometry.

According to the procedure described in section 2.4, we can open the cones $(U_{\rho^2(k)}, ds^2_{(k)}) = \cup_{\alpha=1}^{q(k)} S_\alpha(k)$ and map each conical sector $S_\alpha(k)$ onto a corresponding annulus

$$\Delta_{\vartheta_\alpha(k)}(k) \doteq \{W(k) \in \mathbb{C} \mid e^{-2\pi\vartheta_\alpha(k)} < |W(k)| < 1\}, \tag{54}$$

(see(23),(25)). In such a setting, a natural question to discuss concerns how a deformation of the conical sectors $S_\alpha(k)$, at fixed deficit angle $\varepsilon(k)$ (and perimeter $l(\partial(\rho^2(k))))$, affects the conical geometry of $(U_{\rho^2(k)}, ds^2_{(k)})$, and how this deformation propagates to the underlying Regge polytope $|P_{T_l}| \rightarrow M$. The question is analogous to the study of the non-trivial deformations of constant curvature metrics on a surface. To this end, we adapt to our purposes a standard procedure by considering the following deformation of $S_\alpha(k)$

$$\vartheta_\alpha(k) \longmapsto \vartheta_\alpha(k)' \doteq (s+1)\vartheta_\alpha(k),\ s \in \mathbb{R}, \tag{55}$$

and discuss its effect on the map Υ near the identity $s = 0$. In the annulus description of the geometry of the corresponding sector $S_\alpha(k)$, such a deformation is realized by the quasi-conformal map

$$W(k) \longmapsto f(W(k)) = W(k)\, |W(k)|^s\,, \tag{56}$$

which indeed maps the annulus (54) into the annulus

$$\{W(k) \in \mathbb{C} \mid e^{-2\pi(s+1)\vartheta_\alpha(k)} < |W(k)| < 1\}, \tag{57}$$

corresponding to a conical sector of angle $\vartheta_\alpha(k)' \doteq (s+1)\vartheta_\alpha(k)$. The study of the transformation $f(W(k))$ is well known in the modular theory of the annulus (see [11]), and goes as follows. The Beltrami differential associated with $f(\zeta(k))$ at $s = 0$, is given by

$$\frac{\partial}{\partial \overline{W(k)}} \left[\frac{\partial}{\partial s} f(W(k)) \Big|_{s=0} \right] = \frac{W(k)}{2\overline{W(k)}}. \tag{58}$$

The corresponding Beltrami differential on (54) representing the infinitesimal deformation of $S_\alpha(k)$ in the direction $\partial/\partial(e^{-2\pi\vartheta_\alpha(k)})$ is provided by

$$\begin{aligned} \mu_\alpha(k) \ &\doteq\ \left(\frac{\partial e^{-2\pi(s+1)\vartheta_\alpha(k)}}{\partial s} \right)^{-1}_{s=0} \frac{\partial}{\partial \overline{W(k)}} \left[\frac{\partial}{\partial s} f(W(k)) \Big|_{s=0} \right] = \\ &=\ -\left(\frac{1}{2\pi\vartheta_\alpha(k)} \right) e^{2\pi\vartheta_\alpha(k)} \frac{W(k)}{2\overline{W(k)}}. \end{aligned} \tag{59}$$

Recall that a Beltrami differential on the annulus is naturally paired with a quadratic differential $\widehat{\phi_\alpha(k)} \doteq C_k \frac{dW(k)\otimes dW(k)}{W^2(k)}$, C_k being a real constant, via the L^2 inner product

$$\left\langle \mu_\alpha(k), \widehat{\phi_\alpha(k)} \right\rangle_{\Delta_{\vartheta_\alpha(k)}(k)} \doteq \frac{\sqrt{-1}}{2} \int_{\Delta_{\vartheta_\alpha(k)}(k)} \mu(k)\widehat{\phi_\alpha(k)} dW(k) \wedge d\overline{W(k)}. \tag{60}$$

By requiring that

$$\left\langle \mu_\alpha(k), \widehat{\phi_\alpha(k)} \right\rangle_{\Delta_{\vartheta_\alpha(k)}(k)} = 1 \tag{61}$$

one finds the constant C_k characterizing the quadratic differential $\widehat{\phi_\alpha(k)}$, dual to $\mu_\alpha(k)$, and describing the infinitesimal deformation of $S_\alpha(k)$ in the direction $d(e^{-2\pi\vartheta_\alpha(k)})$, (*i.e.*, a cotangent vector to $\mathfrak{M}_{g,N_0}$ at $((M;N_0),\mathcal{C});\{ds^2_{(k)}\}))$. A direct computation provides

$$\widehat{\phi_\alpha(k)} = -\frac{e^{-2\pi\vartheta_\alpha(k)}}{\pi} \frac{dW(k)\otimes dW(k)}{W^2(k)}. \tag{62}$$

Note that whereas $\phi(k)_{\rho^2(k)}$ may be thought of as a cotangent vector to one of the $\mathbb{R}^{N_0}$ fiber of $\mathfrak{M}_{g,N_0}\times\mathbb{R}^N_+$, the quadratic differential $\widehat{\phi_\alpha(k)}$ projects down to a deformation in the base $\mathfrak{M}_{g,N_0}$, and as such is a much more interesting object than $\phi(k)_{\rho^2(k)}$. The modular nature of $\widehat{\phi_\alpha(k)}$ comes clearly to the fore if we compute its associated Weil-Petersson norm according to

$$\left\| \widehat{\phi_\alpha(k)} \right\|_{W-P} = \int_{\Delta_{\vartheta_\alpha(k)}(k)} \frac{|\widehat{\phi_\alpha(k)}|^2}{g_{hyp}}, \tag{63}$$

where g_{hyp} denotes the hyperbolic metric

$$g_{hyp} \doteq \left(\frac{\pi^2}{\ln^2|t_\alpha(k)|}\right) \frac{dW(k)d\overline{W(k)}}{|W(k)|^2 \sin^2\left(\pi\frac{\ln|W(k)|}{\ln|t_\alpha(k)|}\right)} \tag{64}$$

on the annulus $\Delta_{\vartheta_\alpha(k)}(k)$, and where $|t_\alpha(k)| \doteq e^{-2\pi\vartheta_\alpha(k)}$. A direct computation provides [11]

$$\left\| \widehat{\phi_\alpha(k)} \right\|_{W-P} = |t_\alpha(k)|^2 \left(\frac{1}{\pi}\ln\frac{1}{|t_\alpha(k)|}\right)^3 = 8\vartheta^3_\alpha(k)e^{-4\pi\vartheta_\alpha(k)}. \tag{65}$$

According to proposition 3, the decorated Riemann surface

$$(((M;N_0),\mathcal{C});\{ds^2_{(k)}\})$$

is generated by glueing the oriented domains $(U_{\rho^2(k)}, ds^2_{(k)})$ along the ribbon graph Γ associated with the Regge polytope $|P_{T_l}|\to M$. We can indipendently and holomorphically open the distinct cones $\{(U_{\rho^2(k)}, ds^2_{(k)})\}$ on corresponding sectors $\{S_\alpha(k)\}$, then it follows that to any such $(((M;N_0),\mathcal{C});\{ds^2_{(k)}\})$ we can associate a well defined Weil-Petersson metric which can be immediately read off from (65). Since there are $6g-6+2N_0$ independent conical sectors $S_\alpha(k)$ on the Riemann surface $(((M;N_0),\mathcal{C});\{ds^2_{(k)}\})$ corresponding to the Regge polytope $|P_{T_l}|\to M$, (at fixed deficit angles $\{\varepsilon(k)\}$ and perimeters $\{l(\partial(\rho^2(k)))\}$), we can find a corresponding sequence of angles $\{\vartheta_\alpha(k)\}$ and relabel them as $\{\vartheta_H\}_{H=1}^{6g-6+2N_0}$. If we define

$$\tau_H \doteq e^{-2\pi\vartheta_H} \tag{66}$$

then we can immediately write the Weil-Petersson metric ds^2_{W-P} associated with the Regge polytope $|P_{T_l}| \rightarrow M$

$$\begin{aligned} ds^2_{W-P}(|P_{T_l}|) &= \sum_{H=1}^{6g-6+2N_0} \frac{2\pi^3 d\tau_H^2}{\tau_H^2 \left(\ln \frac{1}{\tau_H}\right)^3} = \\ &= \pi^2 \sum_{H=1}^{6g-6+2N_0} \frac{d\vartheta_H^2}{\vartheta_H^3}. \end{aligned} \tag{67}$$

If we respectively denote by $\theta(H)$ and $l(H)$ the (fixed) conical angle $\theta(k)$ and the perimeter $l(\partial(\rho^2(k)))$ corresponding to the modular variable ϑ_H, (note that the same $\theta(H)$ and $l(H)$ correspond to the distinct $\{\vartheta_H\}$ which are incident on the same vertex), then according to (19) we can rewrite (67) as

$$ds^2_{W-P}(|P_{T_l}|) = \sum_{H=1}^{6g-6+2N_0} \frac{\pi^2}{\theta(H)} \left(\frac{\widehat{L}_H}{l(H)}\right)^{-3} d\left(\frac{\widehat{L}_H}{l(H)}\right) \otimes d\left(\frac{\widehat{L}_H}{l(H)}\right), \tag{68}$$

where $\widehat{L}_H$ is the length variable $\widehat{L}_\alpha(k)$ associated with ϑ_H.

Note that (67) and (68) show a singular behavior when $\vartheta_H \rightarrow 0$ (or equivalently when $\widehat{L}_H \rightarrow 0$), such a behavior occurs for instance when the Regge triangulation (and the corresponding polytope) degenerates and exhibits vertices where thinner and thinner triangles are incident. As recalled in paragraph 2.3, this is usually considered a pathology of Regge calculus. However, in view of the modular correspondence we have described in this paper, it simply corresponds to the well-known incompleteness (with respect of the complex structure of the moduli space) of the Weil-Petersson metric on $\mathfrak{M}_{g,N_0}$ as we approach the boundary (the compactifying divisor) of $\mathfrak{M}_{g,N_0}$ in $\overline{\mathfrak{M}}_{g,N_0}$. In such a sense, as already remarked in the introductory remarks, such a pathology of Regge triangulations has a modular meaning and is not accidental.

4.3 The Weil-Petersson measure on the space of random Regge polytopes.

With the metric $ds^2_{W-P}(|P_{T_l}|)$ we can associate a well defined volume form

$$\Omega_{W-P}(|P_{T_l}|) = \tag{69}$$

$$= \left[\det\left(\frac{\pi^2}{\theta(H)}\left(\frac{\widehat{L}_H}{l(H)}\right)^{-3}\right)\right]^{\frac{1}{2}} d\left(\frac{\widehat{L}_1}{l(1)}\right) \bigwedge \cdots \bigwedge d\left(\frac{\widehat{L}_{6g-6+2N_0}}{l(6g-6+2N_0)}\right),$$

which can be rewritten as a power of product of 2-forms of the type $d\left(\frac{\widehat{L}_H}{l(H)}\right) \wedge d\left(\frac{\widehat{L}_{H+1}}{l(H+1)}\right)$, (this being connected with the Kähler form associated with (68)). Such forms are directly related with the Chern classes (39) of the line bundles

$\mathcal{CL}_k$ and play a distinguished role in the Kontsevich-Witten model. The possibility of expressing the Weil-Petersson Kähler form in terms of the Chern classes (39) is a deep and basic fact of the geometry of $\overline{\mathfrak{M}}_{g,N_0}$, and it is a pleasant feature of the model discussed here that such a connection can be motivated by rather elementary considerations.

The Weil-Petersson volume form $\Omega_{W-P}(|P_{T_l}|)$ allows us to integrate over the space

$$RP^{met}_{g,N_0}(\{\varepsilon(H)\},\{l(H)\}) \tag{70}$$

of distinct Regge polytopes $|P_{T_l}| \to M$ with given deficit angles $\{\varepsilon(k)\}$ and perimeters $\{l(\partial(\rho^2(k)))\}$. To put such an integration in a proper perspective and give it a suggestive physical meaning we shall explicitly consider the set of Regge polytopes

$$RP^{met}_{g,N_0}(\{q(H)\}) \doteq \left\{ |P_{T_l}| \to M|\; \varepsilon(H) = 2\pi - \frac{\pi}{3}q(H);\; l(H) = \frac{\sqrt{3}}{3}aq(H) \right\}, \tag{71}$$

which contain the equilateral Regge polytopes dual to dynamical triangulations. According to proposition 3, $RP^{met}_{g,N_0}(\{q(H)\})$ is a combinatorial description of $\mathfrak{M}_{g,N_0}$, and the volume form $\Omega_{W-P}(|P_{T_l}|)$ can be considered as the pull-back under the morphism Υ, (see (53)) of the Weil-Petersson volume form $\omega_{WP}^{3g-3+N_0}/(3g-3+N_0)!$ on $\mathfrak{M}_{g,N_0}$.

Stated differently, the integration of $\Omega_{W-P}(|P_{T_l}|)$ over the space (71) is equivalent to the Weil-Petersson volume of $\mathfrak{M}_{g,N_0}$. It is well-known that the Weil-Petersson form ω_{WP} extends (as a (1,1) current) to the compactification $\overline{\mathfrak{M}}_{g,N_0}$, and, if we denote by $\overline{RP}^{met}_{g,N_0}(\{q(H)\})$ the compactified orbifold associated with $RP^{met}_{g,N_0}(\{q(H)\})$, then we can write

$$\frac{1}{N_0!}\int_{\overline{RP}^{met}_{g,N_0}(\{q(H)\})} \Omega_{W-P}(|P_{T_l}|) =$$

$$= \frac{1}{N_0!}\int_{\overline{\mathfrak{M}}_{g,N_0}} \frac{\omega_{WP}^{3g-3+N_0}}{(3g-3+N_0)!} = Vol\left(\overline{\mathfrak{M}}_{g,N_0}\right) \tag{72}$$

where we have divided by $N_0(T)!$ in order to factor out the labelling of the $N_0(T)$ punctures. Since $\overline{RP}^{met}_{g,N_0}(\{q(H)\})$ is a (smooth) stratified orbifold acted upon by the automorphism group $Aut_\partial(P_{T_a})$, we can explicitly write the left side of (72) as an orbifold integration over the distinct orbicells Ω_{T_a} (34) in which $\overline{RP}^{met}_{g,N_0}(\{q(H)\})$ is stratified

$$\frac{1}{N_0!}\sum_{T\in\mathcal{DT}[\{q(i)\}_{i=1}^{N_0}]} \frac{1}{|Aut_\partial(P_{T_a})|}\int_{\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})} \Omega_{W-P}(|P_{T_l}|) =$$

$$= VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right), \tag{73}$$

(the orbifold integration is defined in [12], Th. 3.2.1), where the summation is over all distinct dynamical triangulations $\mathcal{DT}[\{q(i)\}_{i=1}^{N_0}]$ with given unlabeled curvature assignments weighted by the order $|Aut_\partial(P_T)|$ of the automorphisms group of the corresponding dual polytope. The relation (73) provides a non-trivial connection between dynamical triangulations (labelling the strata of $\overline{RP}_{g,N_0}^{met}(\{q(H)\})$ (or, equivalently, of $\overline{\mathfrak{M}}_{g,N_0}$), and the fixed connectivity Regge triangulations in each strata Ω_{T_a}. Some aspects of this relation have already been discussed by us elsewhere, [13]. Here we will exploit (73) in order to directly relate $VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right)$ to the partition function of 2D simplicial quantum gravity. To this end let us sum both members of (73) over the set of all possible curvature assignments $\{q(H)\}$ on the N_0 unlabelled vertices of the triangulations, and note that

$$Card\left[\mathcal{DT}(N_0)\right] \doteq \sum_{\mathcal{DT}(N_0)} \frac{1}{|Aut_\partial(P_{T_a})|} = \\ = \frac{1}{N_0!} \sum_{\{q(H)\}_{H=1}^{N_0}} \sum_{T\in\mathcal{DT}[\{q(H)\}_{H=1}^{N_0}]} \frac{1}{|Aut_\partial(P_{T_a})|}$$

provides the number of distinct (generalized) dynamical triangulations with N_0 unlabelled vertices. Since $VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right)$ does not depend of the curvature assignments $\{q(H)\}$, from (73) we get

$$\frac{1}{N_0!} \sum_{\{q(H)\}_{H=1}^{N_0}} \sum_{T\in\mathcal{DT}[\{q(i)\}_{i=1}^{N_0}]} \frac{1}{|Aut_\partial(P_{T_a})|} \int_{\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})} \Omega_{W-P}(|P_{T_l}|) = \\ = \sum_{\mathcal{DT}(N_0)} \frac{1}{|Aut_\partial(P_{T_a})|} \int_{\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})} \Omega_{W-P}(|P_{T_l}|) = \\ = \left(Card\{q(H)\}\right) VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right), \tag{74}$$

where $Card\{q(H)\}$ denotes the number of possible curvature assignments on the N_0 unlabelled vertices of the triangulations. Dividing both members by $(Card\{q(H)\})$, we eventually get the relation

$$\sum_{\mathcal{DT}(N_0)} \frac{1}{|Aut_\partial(P_{T_a})|} \int_{\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})} \frac{\Omega_{W-P}(|P_{T_l}|)}{Card\{q(H)\}} = VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right), \tag{75}$$

(the number $Card\{q(H)\}$ has been shifted under the integral sign for typographical convenience). We have the following

Lemma 4 *For N_0 sufficiently large there exist a positive constant $K(g)$ independent from N_0 but possibly dependent on the genus g, and a positive constant κ independent both from N_0 and g such that*

$$\int_{\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})} \frac{\Omega_{W-P}(|P_{T_l}|)}{Card\{q(H)\}} \simeq K(g)e^{-\kappa N_0}. \tag{76}$$

In order to prove this result let us start by recalling that the large $N_0(T)$ asymptotics of the triangulation counting $Card\,[\mathcal{DT}[N_0]]$ can be obtained from purely combinatorial (and matrix theory) arguments [2], [14] to the effect that

$$Card\,[\mathcal{DT}[N_0]] \sim \frac{16c_g}{3\sqrt{2\pi}} \cdot e^{\mu_0 N_0(T)} N_0(T)^{\frac{5g-7}{2}} \left(1 + O(\frac{1}{N_0})\right), \tag{77}$$

where c_g is a numerical constant depending only on the genus g, and $e^{\mu_0} = (108\sqrt{3})$ is a (non-universal) parameter depending on the set of triangulations considered (here the generalized triangulations, barycentrically dual to trivalent graphs; in the case of regular triangulations in place of $108\sqrt{3}$ we would get $e^{\mu_0} = (\frac{4^4}{3^3})$). Thus, if we denote by

$$\left\langle \int \frac{\Omega_{W-P}(|P_{T_l}|)}{Card\{q(H)\}} \right\rangle_{\mathcal{DT}[N_0]} \doteq$$
$$= \frac{1}{Card\,[\mathcal{DT}[N_0]]} \sum_{\mathcal{DT}(N_0)} \frac{1}{|Aut_\partial(P_{T_a})|} \int_{\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})} \frac{\Omega_{W-P}(|P_{T_l}|)}{Card\{q(H)\}} \tag{78}$$

the average value of $\int \Omega_{W-P}(|P_{T_l}|)/Card\{q(H)\}$ over the set $\mathcal{DT}[N_0]$, (dropping the integration range $\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})$ for notational ease), then we can write the large N_0 asymptotics of the left side member of (73) as

$$\sum_{\mathcal{DT}(N_0)} \frac{1}{|Aut_\partial(P_{T_a})|} \int_{\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})} \frac{\Omega_{W-P}(|P_{T_l}|)}{Card\{q(H)\}} \simeq$$
$$\simeq \frac{16c_g}{3\sqrt{2\pi}} \left\langle \int \frac{\Omega_{W-P}(|P_{T_l}|)}{Card\{q(H)\}} \right\rangle_{\mathcal{DT}[N_0]} e^{\mu_0 N_0(T)} N_0(T)^{\frac{5g-7}{2}} \left(1 + O(\frac{1}{N_0})\right). \tag{79}$$

On the other hand, from the Manin-Zograf asymptotic analysis of $VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right)$ for fixed genus g and large N_0, we have[1][15]

$$VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right) =$$
$$= \pi^{2(3g-3+N_0)}(N_0+1)^{\frac{5g-7}{2}} C^{-N_0} \left(B_g + \sum_{k=1}^{\infty} \frac{B_{g,k}}{(N_0+1)^k}\right), \tag{80}$$

where $C = -\frac{1}{2} j_0 \frac{d}{dz} J_0(z)|_{z=j_0}$, ($J_0(z)$ the

Bessel function, j_0 its first positive zero); (note that $C \simeq 0.625....$). The genus dependent parameters B_g are explicitly given [15] by

$$\begin{cases} B_0 = \frac{1}{A^{1/2}\Gamma(-\frac{1}{2})C^{1/2}}, & B_1 = \frac{1}{48}, \\ B_g = \frac{A^{\frac{g-1}{2}}}{2^{2g-2}(3g-3)!\Gamma(\frac{5g-5}{2})C^{\frac{5g-5}{2}}} \left\langle \tau_2^{3g-3} \right\rangle, & g \geq 2 \end{cases} \tag{81}$$

where $A \doteq -j_0^{-1} J_0'(j_0)$, and $\left\langle \tau_2^{3g-3} \right\rangle$ is a Kontsevich-Witten intersection number [9][16], (the coefficients $B_{g,k}$ can be computed similarly-see [15] for details).

By Inserting (80) in the left hand side of (73) and by taking into account (4.3) we eventually get

$$\frac{16c_g}{3\sqrt{2\pi}}\left\langle\int\frac{\Omega_{W-P}(|P_{T_l}|)}{Card\{q(H)\}}\right\rangle_{\mathcal{DT}[N_0]}e^{\mu_0 N_0(T)}N_0(T)^{\frac{5g-7}{2}}\left(1+O(\frac{1}{N_0})\right)= \tag{82}$$
$$=\pi^{2(3g-3+N_0)}(N_0+1)^{\frac{5g-7}{2}}C^{-N_0}\left(B_g+\sum_{k=1}^{\infty}\frac{B_{g,k}}{(N_0+1)^k}\right),$$

which by direct comparison provides

$$\left\langle\int\frac{\Omega_{W-P}(|P_{T_l}|)}{Card\{q(H)\}}\right\rangle_{\mathcal{DT}[N_0]}\simeq\frac{3\sqrt{2\pi}B_g\pi^{2(3g-3+N_0)}}{16c_g}e^{(|\ln C|-\mu_0)N_0}. \tag{83}$$

This proves the lemma with

$$K(g)=\frac{3\sqrt{2\pi}B_g\pi^{2(3g-3)}}{16c_g}, \tag{84}$$

and

$$\kappa=\mu_0-|\ln C|-2\ln\pi\approx 2.472. \tag{85}$$

Thus, according to the above lemma, we can rewrite the left member of (75) as

$$VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right)=\sum_{\mathcal{DT}(N_0)}\frac{1}{|Aut_\partial(P_{T_a})|}\int_{\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})}\frac{\Omega_{W-P}(|P_{T_l}|)}{Card\{q(H)\}}= \tag{86}$$
$$=K(g)\sum_{\mathcal{DT}(N_0)}\frac{1}{|Aut_\partial(P_{T_a})|}e^{-\kappa N_0},$$

which has the structure of the canonical partition function for dynamical triangulation theory [6]. Roughly speaking, we may interpret (86) by saying that dynamical triangulations count the (orbi)cells which contribute to the volume (each cell containing the Regge triangulations whose adjacency matrix is that one of the dynamical triangulation which label the cell), whereas integration over the cells weights the fluctuations due to all Regge triangulations which, in a sense, represent the deformational degrees of freedom of the given dynamical triangulation labelling the cell.

5 Concluding remarks

Note that in the canonical partition function of 2D simplicial quantum gravity, $-\kappa$ has the role of a running coupling constant (a chemical potential in a grand-canonical description), whereas in (86) it takes on the value (85). This remark suggests that $e^{\lambda N_0}VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right)$ with λ a running coupling constant is the canonical partition function of 2D simplicial quantum gravity. More generally,

if we let also fluctuate the topology of the triangulated surfaces $|T_l| \to M$ then we have strong elements suggesting that the canonical partition function of 2D simplicial quantum gravity over the set of (orientable) surfaces with random topology is

$$e^{\lambda N_0 + \eta g} VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right). \tag{87}$$

In this connection it is also important to stress that (80) shows very naturally how the genus-g pure gravity critical exponent

$$\gamma_g = \frac{5g-1}{2}. \tag{88}$$

originates from the cardinality of the cell decomposition of $\overline{\mathfrak{M}}_{g,N_0}$ parametrized by $T \in \mathcal{DT}[\{q(i)\}_{i=1}^{N_0}]$, (this remark also explains geometrically why standard, fixed connectivity Regge calculus, is not able to recover γ_g).

As already argued in [3] on a slightly different basis, we have a natural candidate which should play in the matter setting the same role of $\overline{\mathfrak{M}}_{g,N_0}$ in the pure gravity case. This is the space $\overline{\mathfrak{M}}_{g,N_0}(X,\beta)$ of stable maps from $((M;N_0),\mathcal{C})$ to the manifold X parametrizing the matter fields and representing the homology class $\beta \in H_2(X,\mathbb{Z})$, (for instance, in the case of the Polyakov action, the map can be thought of as providing the embedding of $((M;N_0),\mathcal{C})$ in the target (Euclidean) space). According to the above remarks, it is rather natural to conjecture that the canonical partition function describing conformal matter interacting with 2D simplicial quantum gravity is provided by a (generalized) Weil-Petersson volume of $\overline{\mathfrak{M}}_{g,N_0}(X,\beta)$. Candidates for such volumes are related to the (descendent) Gromov-Witten invariants of the manifold X, and evidence in such a direction comes from the well-known fact that many relevant properties of $\overline{\mathfrak{M}}_{g,N_0}(X,\beta)$ are governed, via its intersection theory, by matrix models [17][18]. Bringing such a program to completion appear a rather formidable task, however it will certainly throw new light on the deep nature of simplicial techniques, Regge calculus and their foundational role in disclosing the physics of quantum gravity.

Acknowledgements

This work was supported in part by the Ministero dell'Universita' e della Ricerca Scientifica under the PRIN project *The geometry of integrable systems.*

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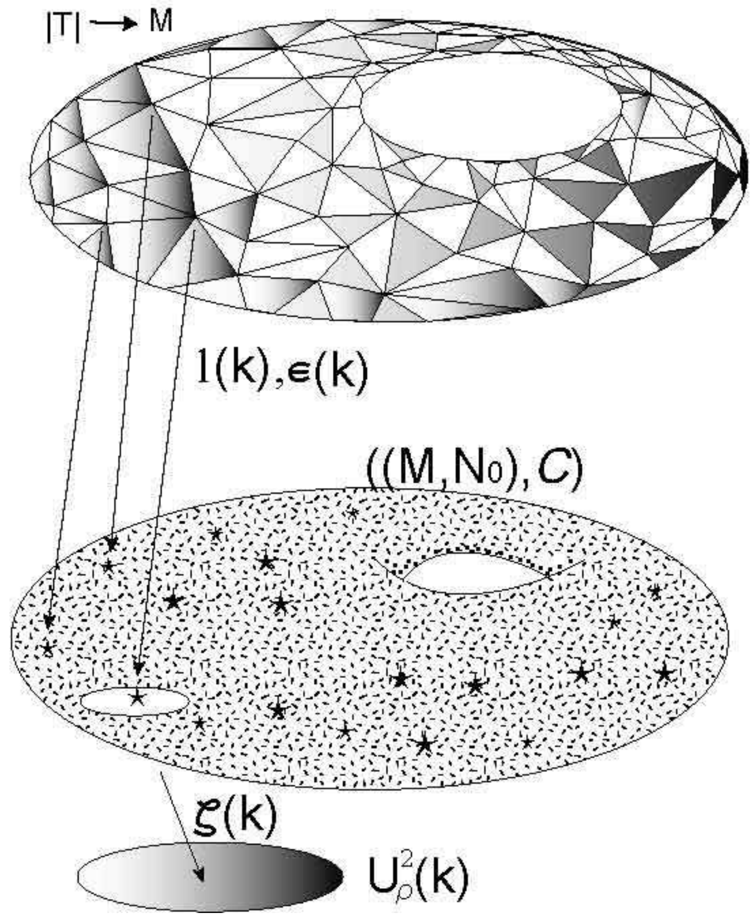

Figure 1: A torus triangulated with triangles of variable edge-length.

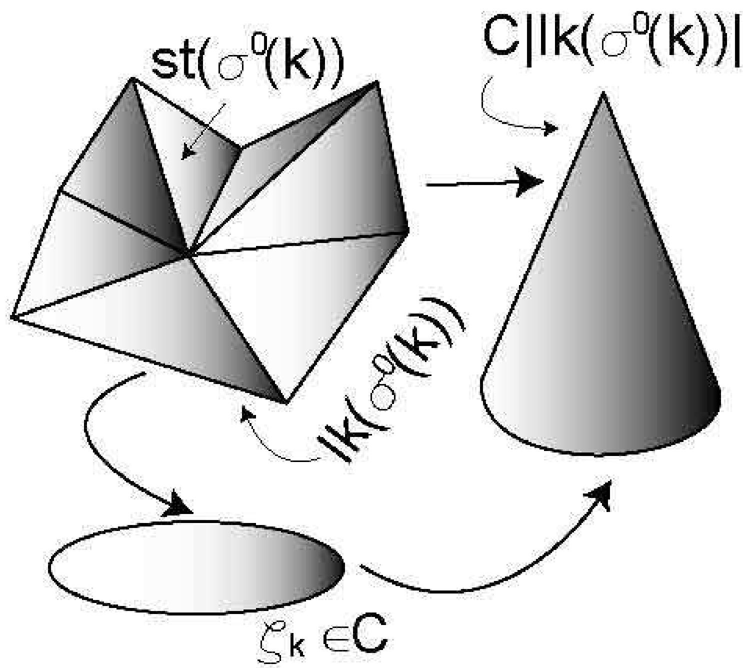

Figure 2: The geometric structures around a vertex.

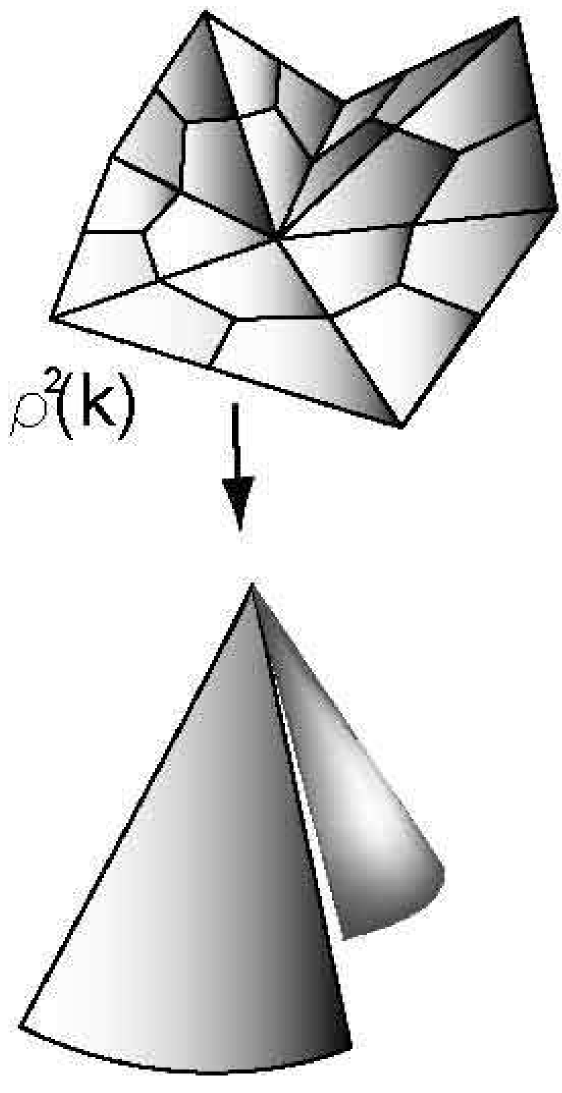

Figure 3: The conical geometry of the baricentrically dual polytope.

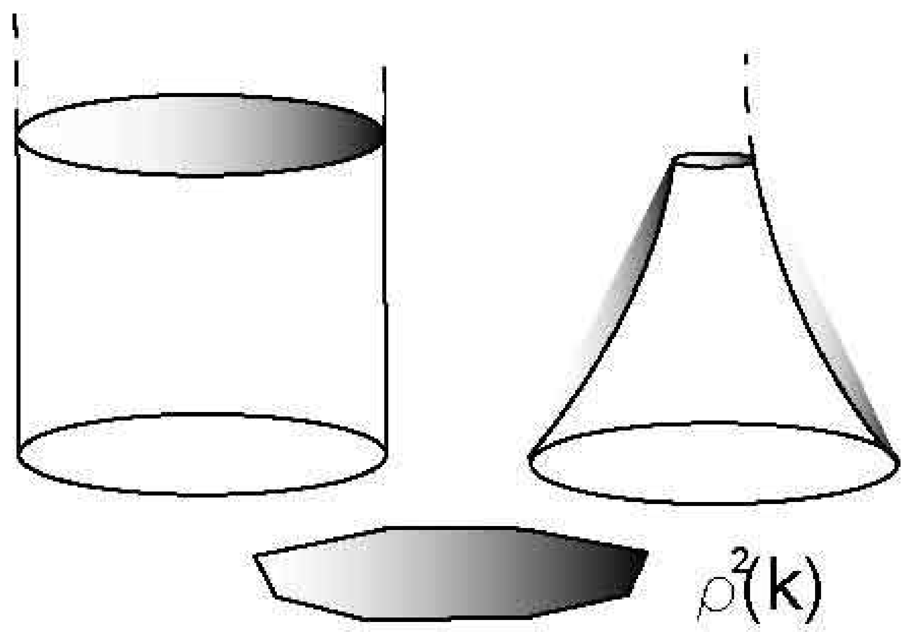

Figure 4: The cylindrical and hyperbolic metric over a $\theta \to 0$ degenerating polytopal cell.

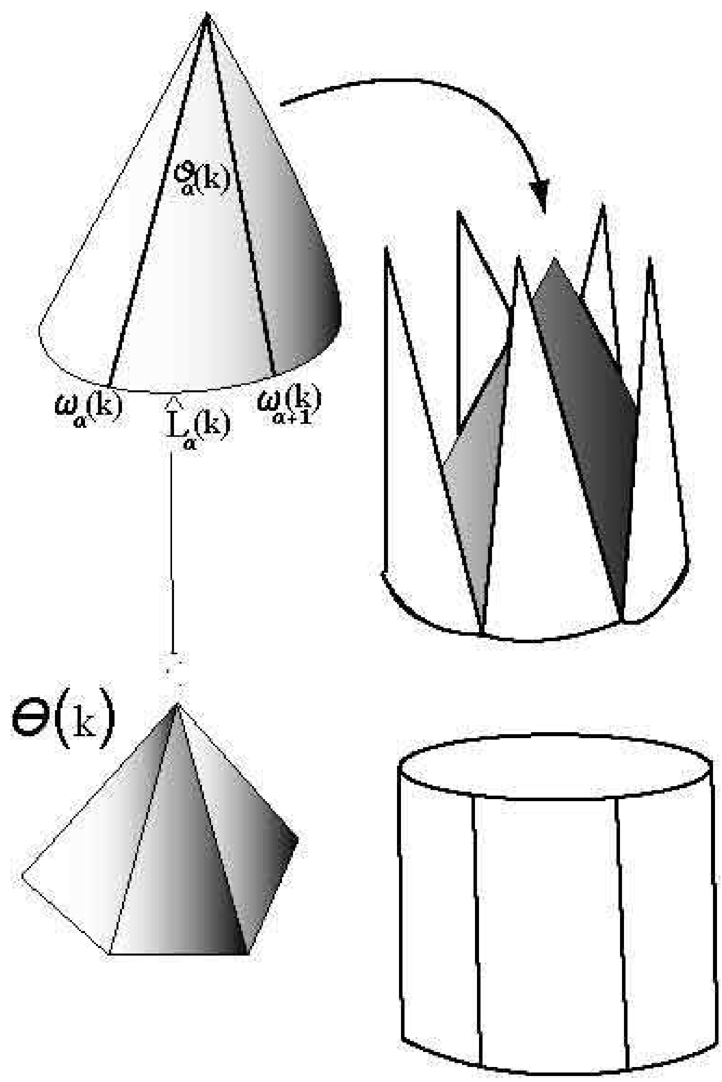

Figure 5: The opening of the cone into its constituents conical sectors and the associated cylindrical strip.

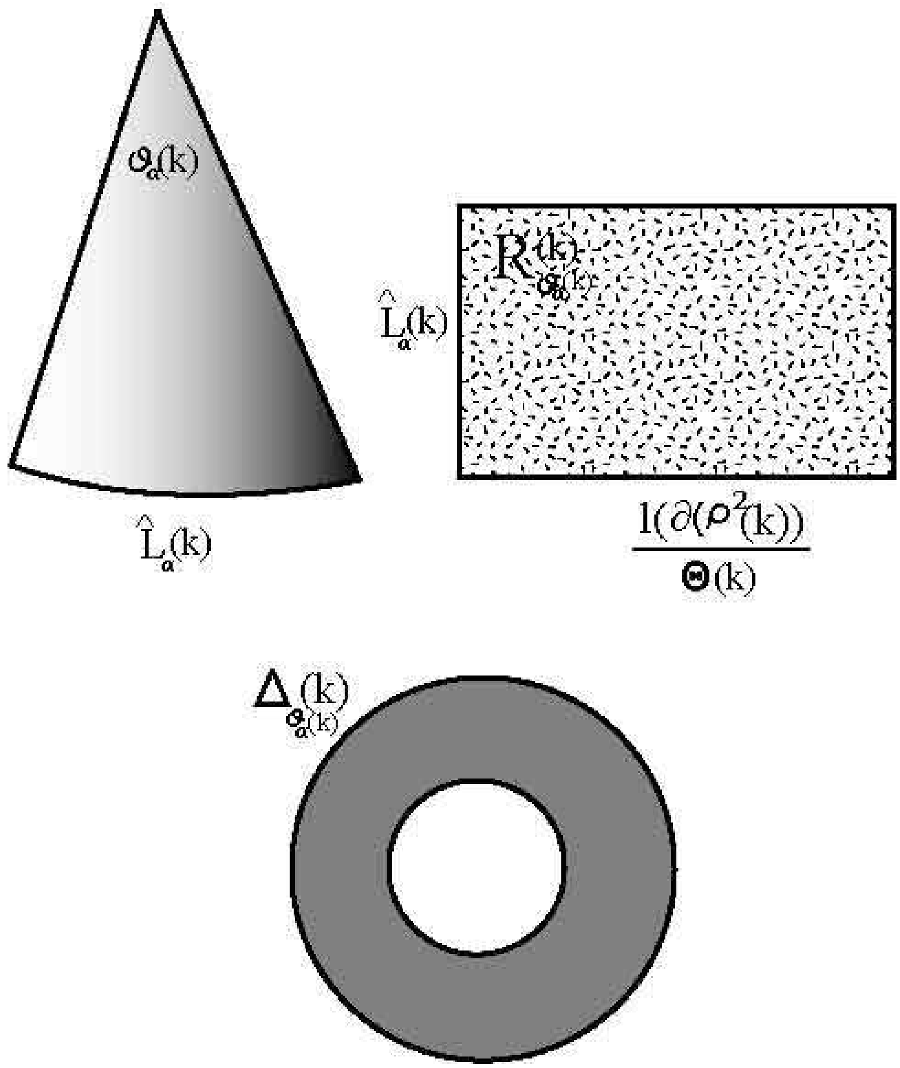

Figure 6: The mapping of a conical sector in a strip and into an anular region.

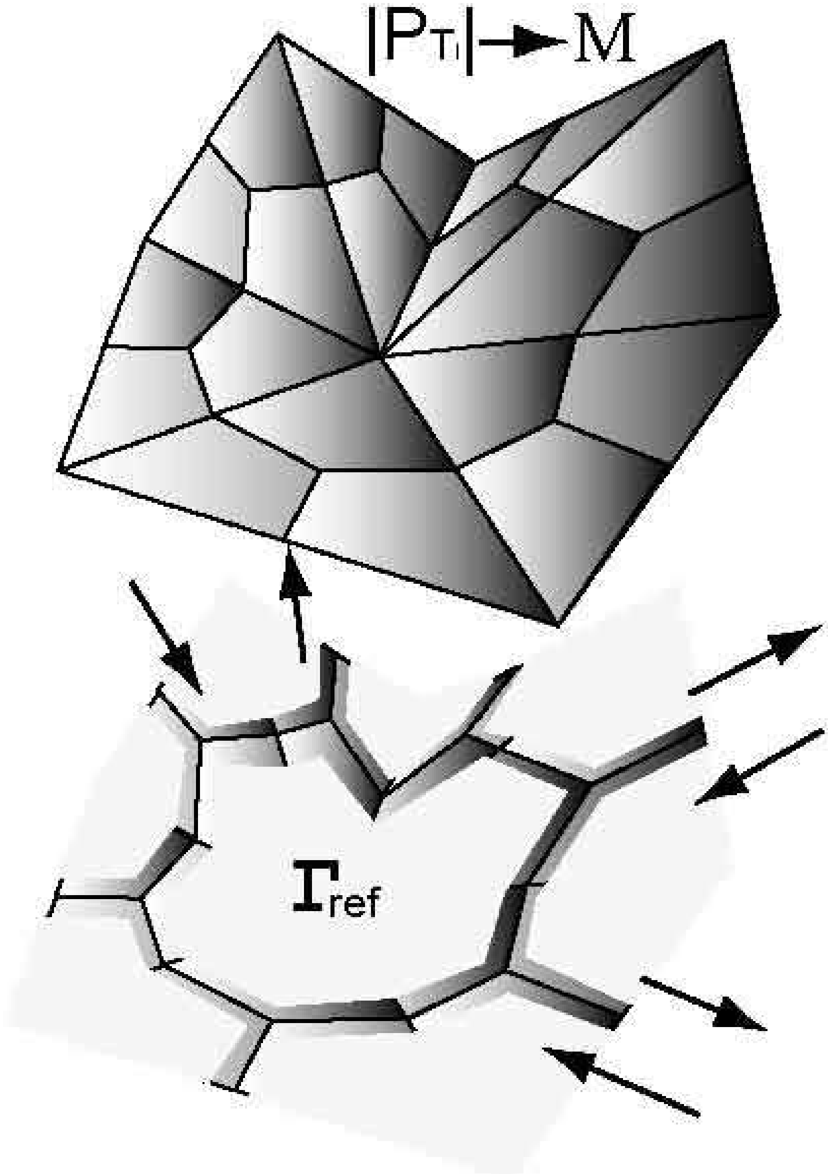

Figure 7: The dual polytope around a vertex and it's edge refinement.

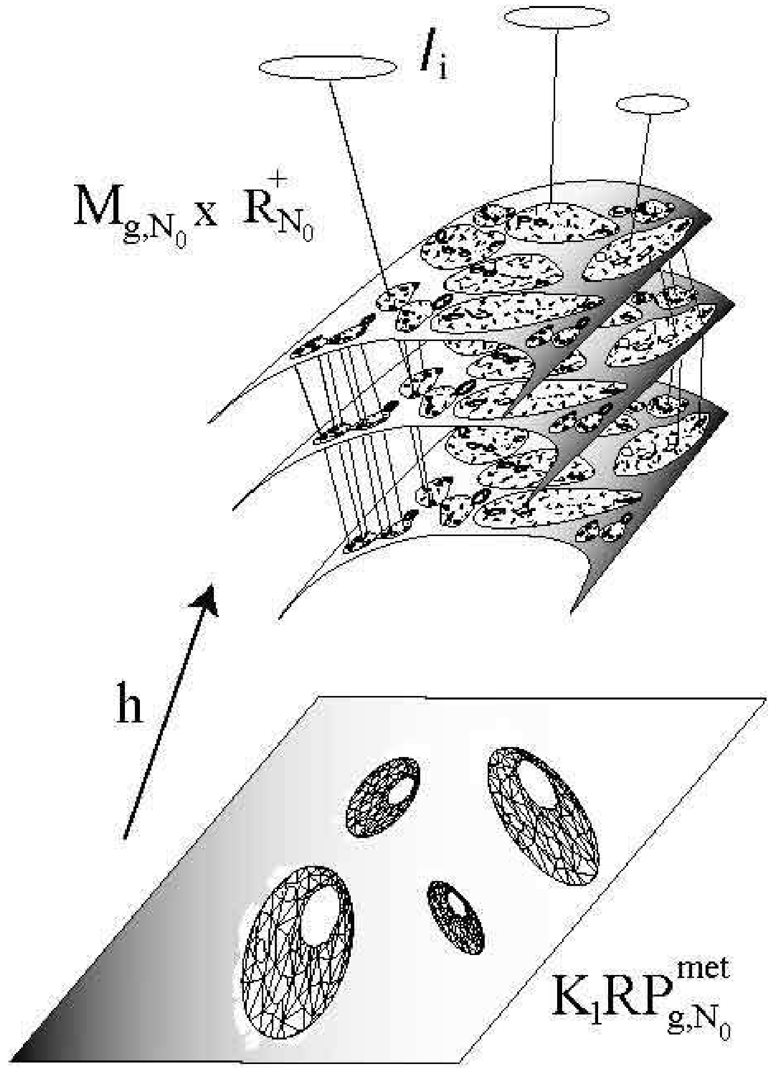

Figure 8: The map h associates to each ribbon graph an element of the decorated moduli space $\mathfrak{M}_{g,N_0} \times \Re^+_{N_0}$.

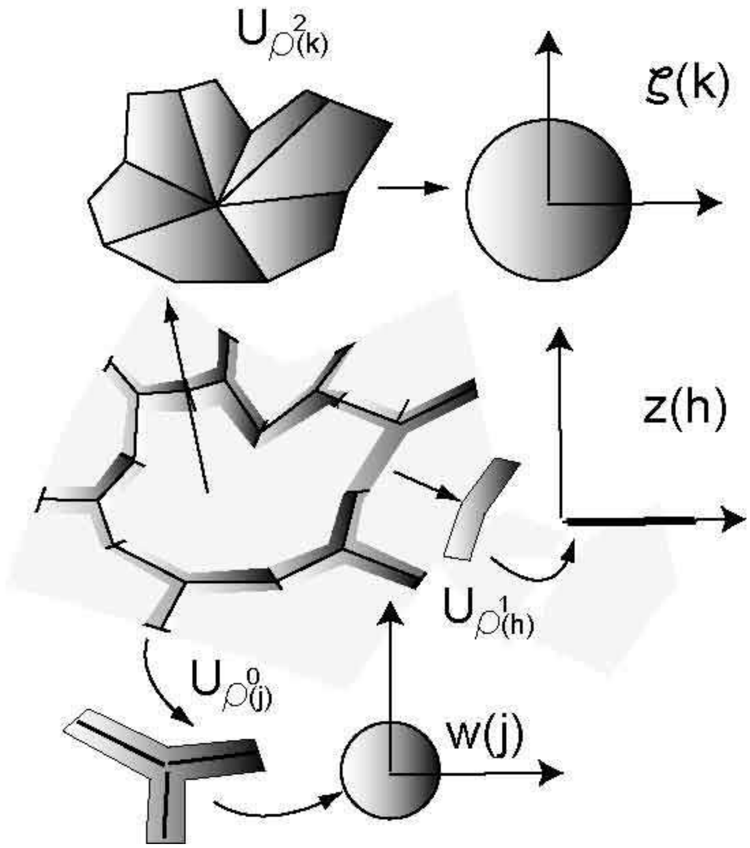

Figure 9: The local presentation of uniformizing coordinates.

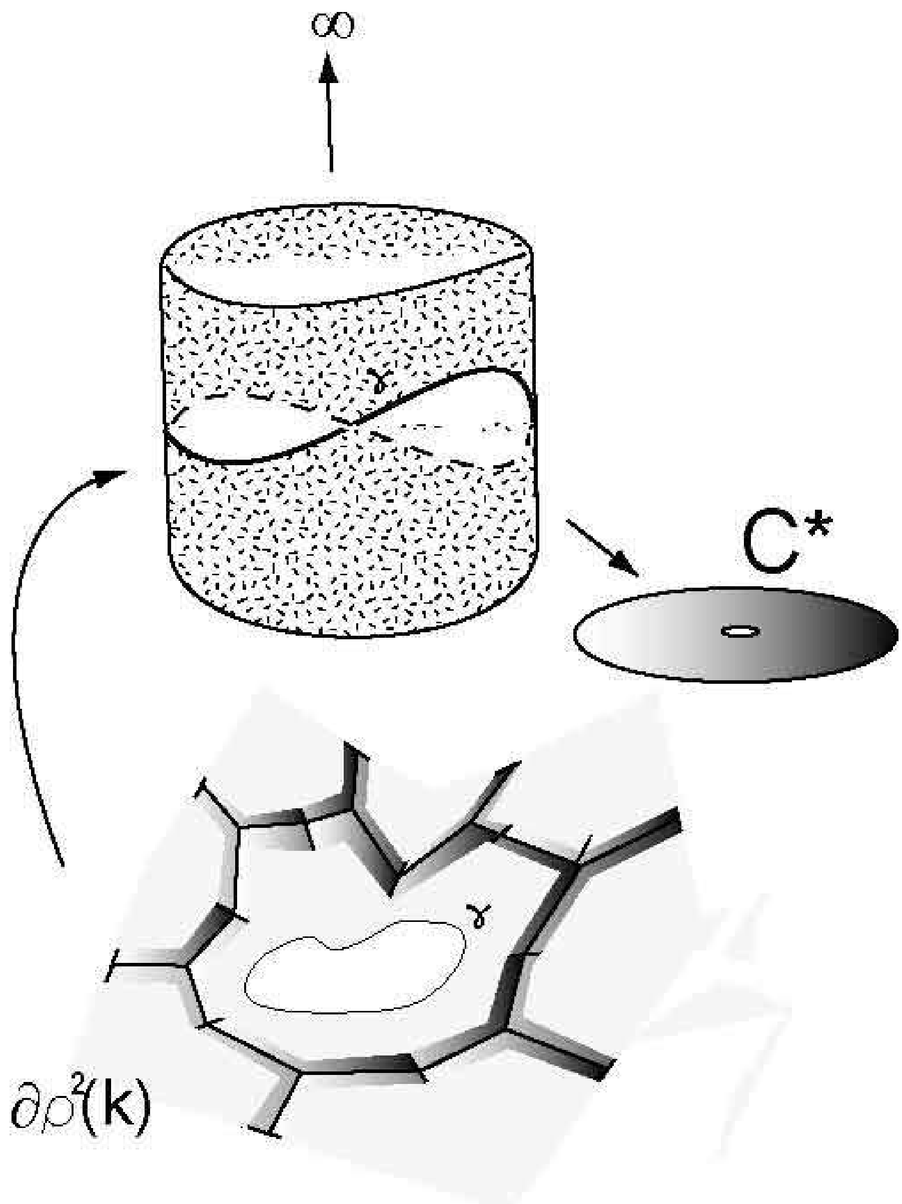

Figure 10: The cylindrical metric associated with a ribbon graph.

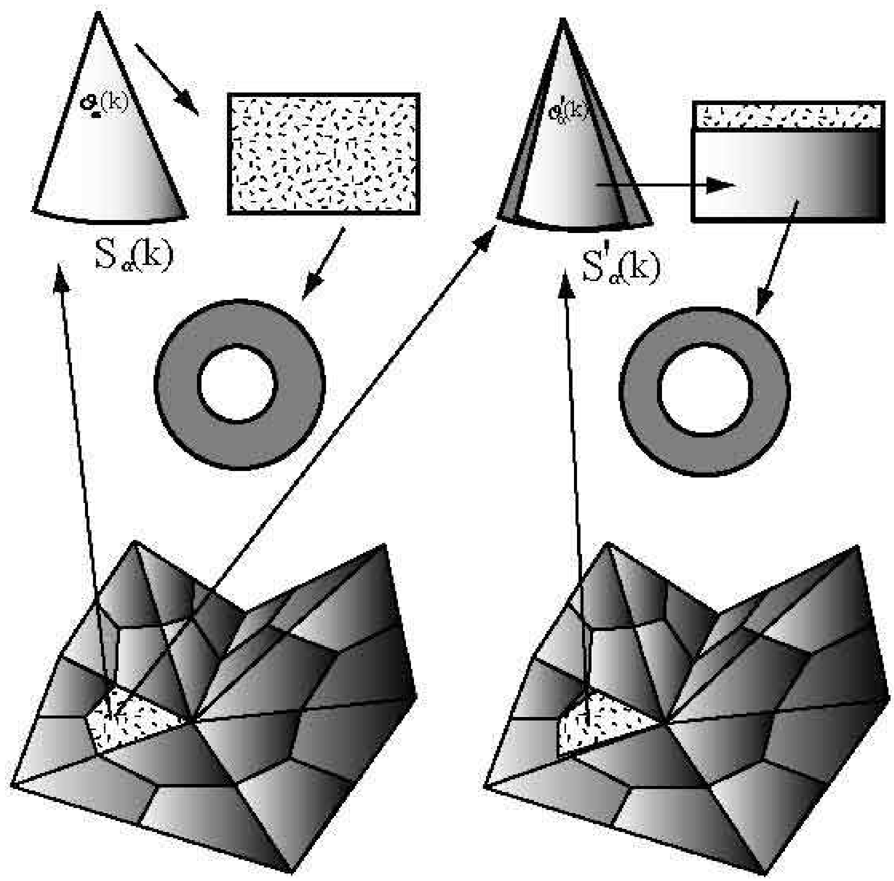

Figure 11: The modular deformation of a conical sector in a Regge polytope.